Laser Wakefield Accelerators

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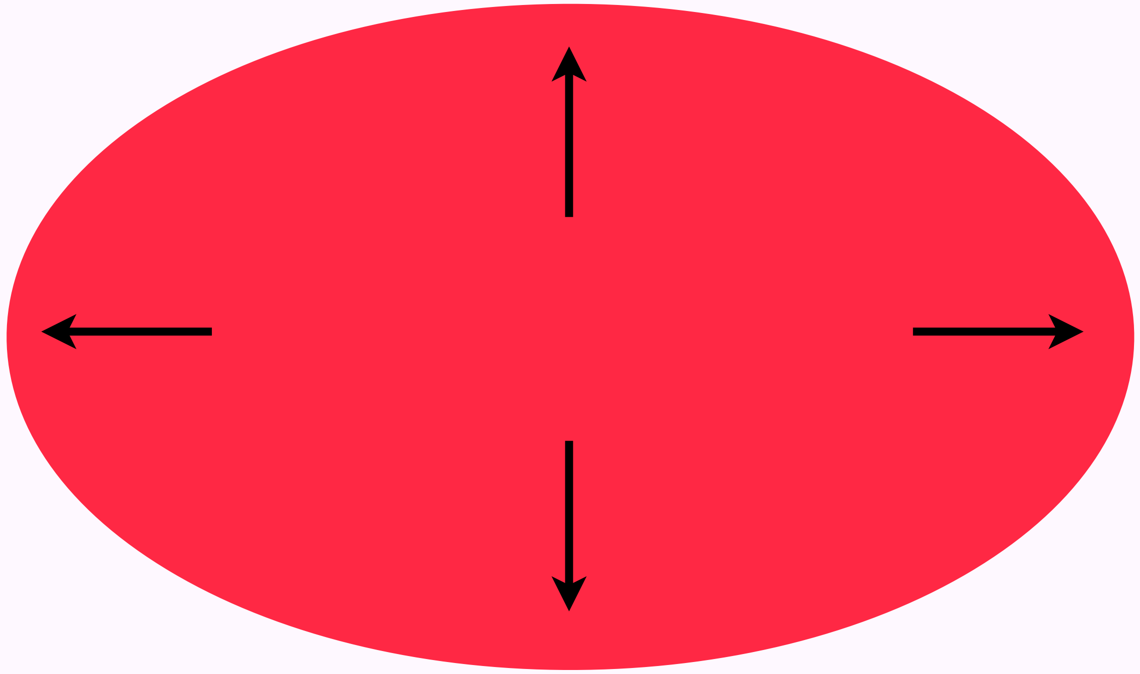


$$\frac{\partial \mathbf{p}}{\partial t} = -mc^2 \nabla \gamma \qquad \text{(Ponderomotive)}$$

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -\mathbf{v} \times \mathbf{B} = -\frac{\mathbf{p}}{\gamma} \times (\nabla \times \mathbf{a})$$

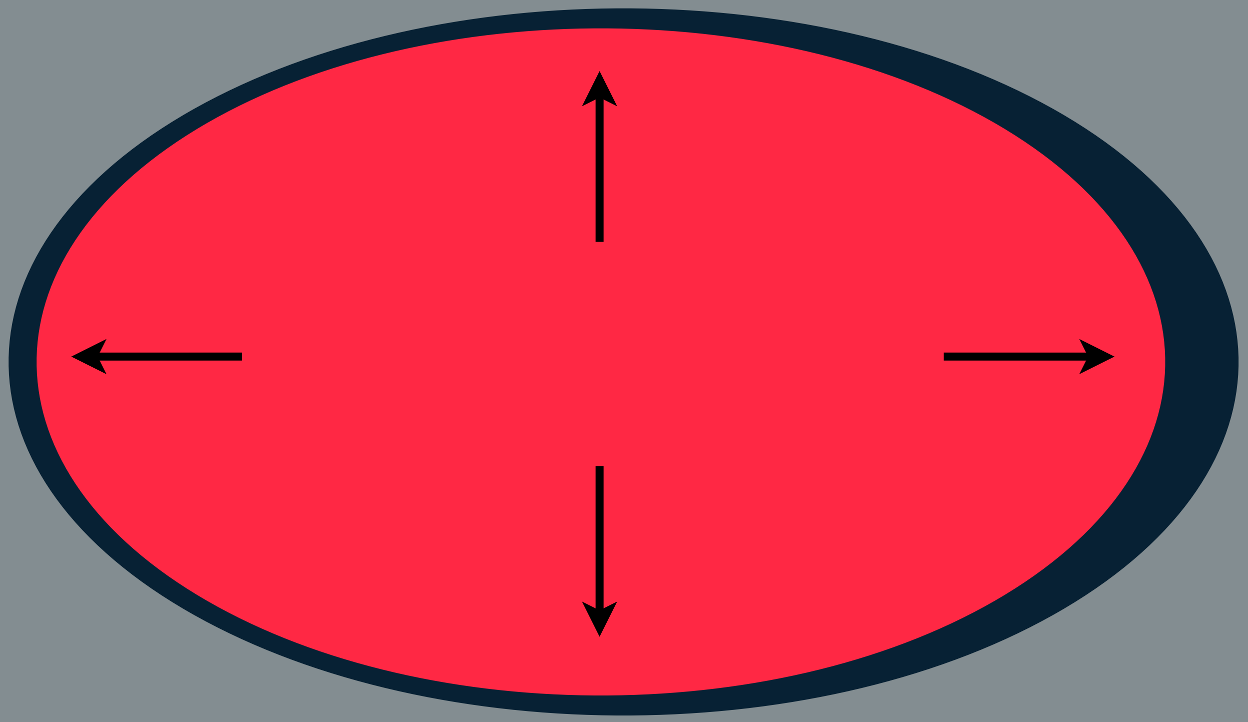
$$\frac{\partial \mathbf{p}}{\partial t} + (\mathbf{v} \cdot \nabla)\mathbf{p} = -\frac{1}{\gamma} (\nabla(\mathbf{p} \cdot \mathbf{a}) - (\mathbf{p} \cdot \nabla)\mathbf{a})$$

$$\frac{\partial \mathbf{p}}{\partial t} = -\frac{1}{\gamma} \left(\nabla a^2 - (\mathbf{p} \cdot \nabla)(\mathbf{a} - \mathbf{p})\right)$$

$$\frac{\partial \mathbf{p}}{\partial t} = -\frac{1}{\gamma} \nabla a^2 - \frac{1}{\gamma} \nabla p_x^2$$

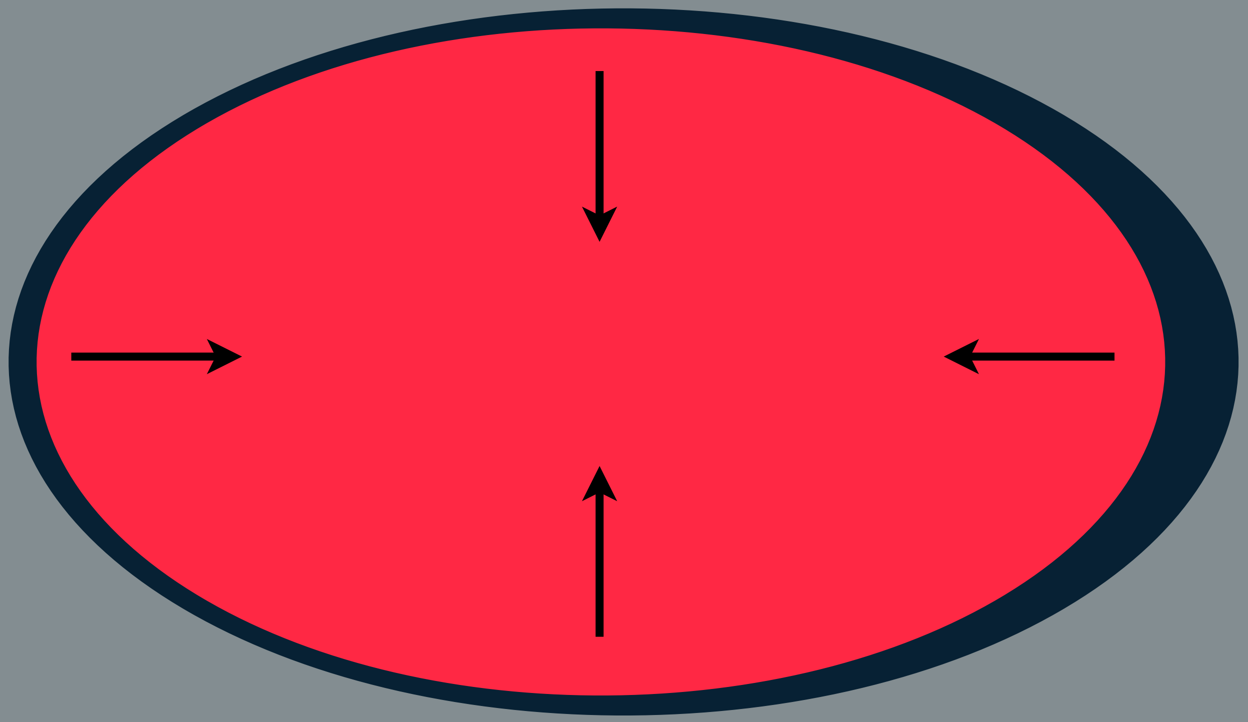
with $\gamma = \sqrt{1 + {p_x}^2 + a^2}$

$$\rightarrow \frac{\partial \mathbf{p}}{\partial t} = -\nabla \gamma$$



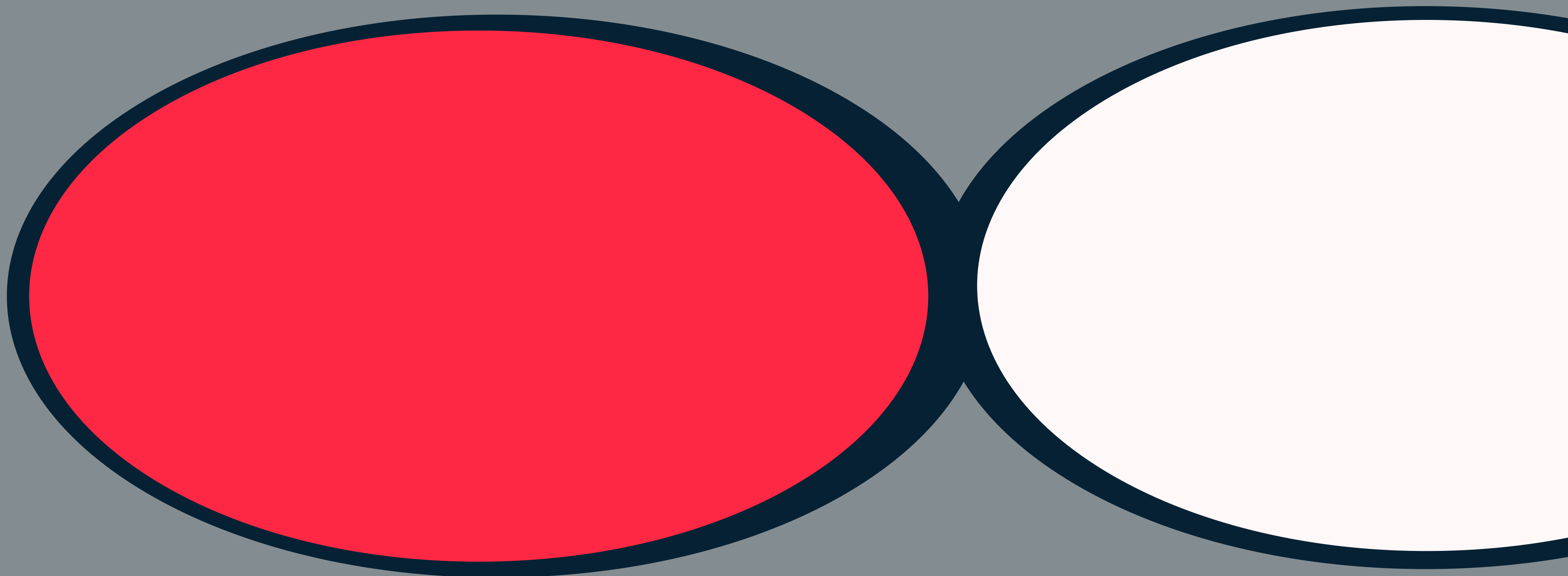
$$\frac{\partial \mathbf{p}}{\partial t} = -mc^2 \nabla \gamma$$

(Ponderomotive)



$$\frac{\partial \mathbf{p}}{\partial t} = -e\mathbf{E} - mc^2\nabla\gamma \qquad \text{(motion)}$$

$$\nabla \cdot \mathbf{E} = -e(n_e - n_i)/\epsilon_0 \qquad \text{(Gauss)}$$



$$\frac{\partial \mathbf{p}}{\partial t} = -e\mathbf{E} - mc^2\nabla\gamma \qquad \text{(motion)}$$

$$\nabla \cdot \mathbf{E} = -e(n_e - n_i)/\epsilon_0 \qquad \text{(Gauss)}$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e v) = 0 \qquad \text{(continuity)}$$

normalised units : $e = 1;\ m_e = 1;\ c = 1;\ \epsilon_0 = 1$

$$\frac{\partial \mathbf{p}}{\partial t} = -\mathbf{E} - \nabla\gamma \qquad \text{(motion)}$$

$$\nabla \cdot \mathbf{E} = -(n_e - n_i) \qquad \text{(Gauss)}$$

$$\frac{\partial n_e}{\partial t} + \nabla \cdot (n_e v) = 0 \qquad \text{(continuity)}$$

one dimensional : $E = E_x;\ p = p_x = \gamma\beta$

also take $n_i = n_0$

$$\frac{\partial p}{\partial t} = -E - \frac{\partial \gamma}{\partial x} \qquad \text{(motion)}$$

$$\frac{\partial E}{\partial x} = (n_0 - n_e) \qquad \text{(Gauss)}$$

$$\frac{\partial n_e}{\partial t} + \frac{\partial}{\partial x}(n_e v) = 0 \qquad \text{(continuity)}$$

Quasistatic approximation : $\xi = x - t$

$$\frac{\partial}{\partial t} = \frac{\partial \xi}{\partial t}\frac{\partial}{\partial \xi} = -\frac{\partial}{\partial \xi}; \quad \frac{\partial}{\partial t} = \frac{\partial \xi}{\partial x}\frac{\partial}{\partial \xi} = \frac{\partial}{\partial \xi}$$

$$-\frac{\partial p}{\partial \xi} = \frac{\partial \phi}{\partial \xi} - \frac{\partial \gamma}{\partial \xi} \tag{1}$$

$$\frac{\partial^2 \phi}{\partial \xi^2} = (n_e - n_0) \tag{2}$$

$$-\frac{\partial n_e}{\partial \xi} + \frac{\partial (n_e \beta)}{\partial \xi} = 0 \tag{3}$$

$$-\frac{\partial p}{\partial \xi} = \frac{\partial \phi}{\partial \xi} - \frac{\partial \gamma}{\partial \xi} \qquad (1)$$

$$\frac{\partial^2 \phi}{\partial \xi^2} = (n_e - n_0) \qquad (2)$$

$$-\frac{\partial n_e}{\partial \xi} + \frac{\partial (n_e \beta)}{\partial \xi} = 0 \qquad (3)$$

From (3) : $n_e(1-\beta) = \text{constant} = n_0$

so $\dfrac{\partial^2 \phi}{\partial \xi^2} = n_0 \left(1 - \dfrac{1}{1-\beta}\right)$ (Poisson)

NB since $-1 < \beta < 1$: $\qquad \dfrac{1}{2} < n_e < \infty$

$$-\frac{\partial p}{\partial \xi} = \frac{\partial \phi}{\partial \xi} - \frac{\partial \gamma}{\partial \xi} \qquad (1)$$

$$\frac{\partial^2 \phi}{\partial \xi^2} = n_0 \left(1 - \frac{1}{1 - \beta} \right)$$

$$\text{From (1)} : \gamma - \beta\gamma - \phi = \text{constant} = 1$$

$$\rightarrow \quad \gamma(1 - \beta) = 1 + \phi$$

$$\text{Also from (1)} : \gamma^2 = 1 + p^2 + a^2$$

$$\rightarrow \gamma^2(1 - \beta^2) = 1 + a^2$$

$$\text{Eliminating } \beta : \ \frac{\partial^2 \phi}{\partial \xi^2} = -\frac{1}{2} {k_p}^2 \left(\frac{1 + a^2}{(1 + \phi)^2} - 1 \right)$$

Wakefield generation

$$\frac{1}{{k_p}^2}\frac{\partial^2\phi}{\partial\zeta^2} = \frac{n_b}{n_0} + {\gamma_p}^2\left\{\beta_p\left[1 - \frac{(1+a^2)}{{\gamma_p}^2(1+\phi)^2}\right]^{-1/2} - 1\right\}$$

quasistatic

beam

laser

Non-linear **quasistatic** 1D wakefield equation.
Can be simplified by noting $\gamma_p \to 1$ in most cases.

Taking $\gamma_p \gg 1$, $\beta_p = (1 - \frac{1}{{\gamma_p}^2})^{1/2} \approx (1 - \frac{1}{2}\frac{1}{{\gamma_p}^2})$

and expanding the square bracket gives:

$$\frac{1}{{k_p}^2}\frac{\partial^2\phi}{\partial\zeta^2} = \frac{1}{2}\left[\frac{(1+a^2)}{(1+\phi)^2} - 1\right]$$

Wakefield generation

$$\frac{1}{{k_p}^2}\frac{\partial^2\phi}{\partial\zeta^2} = \frac{1}{2}\left[\frac{(1+a^2)}{(1+\phi)^2} - 1\right]$$

Take $\phi \ll 1$,

$$\left(\frac{\partial^2}{\partial\zeta^2} + {k_p}^2\right)\phi = \frac{1}{2}{k_p}^2 a^2$$

A simple force oscillator. Take for example a laser intensity profile that goes as:

$$a^2 = {a_0}^2 \sin^2(\pi\zeta/L)$$

Wakefield generation

Solving (in 1D):

$$\frac{\partial E}{\partial \zeta} = -n_1 \qquad \text{(Gauss' Law)}$$

$$\frac{\partial n_1}{\partial \zeta} = \frac{\partial (n_e \beta)}{\partial \zeta} \qquad \text{(Continuity)}$$

$$(1-\beta)\frac{\partial \beta}{\partial \zeta} = eE - \frac{1}{\gamma}\frac{\partial (a^2)}{\partial \zeta} \qquad \text{(Motion)}$$

where $\beta = v/c$, $n_1 = \delta n/n_0$, and $E = E_{wf}/E_0$

(or alternatively m_e, c, ϵ_0, c all normalised to 1).

Wakefield generation

Solving (in 1D):

$$\frac{\partial E}{\partial \zeta} = -n_1 \qquad \text{(Gauss' Law)}$$

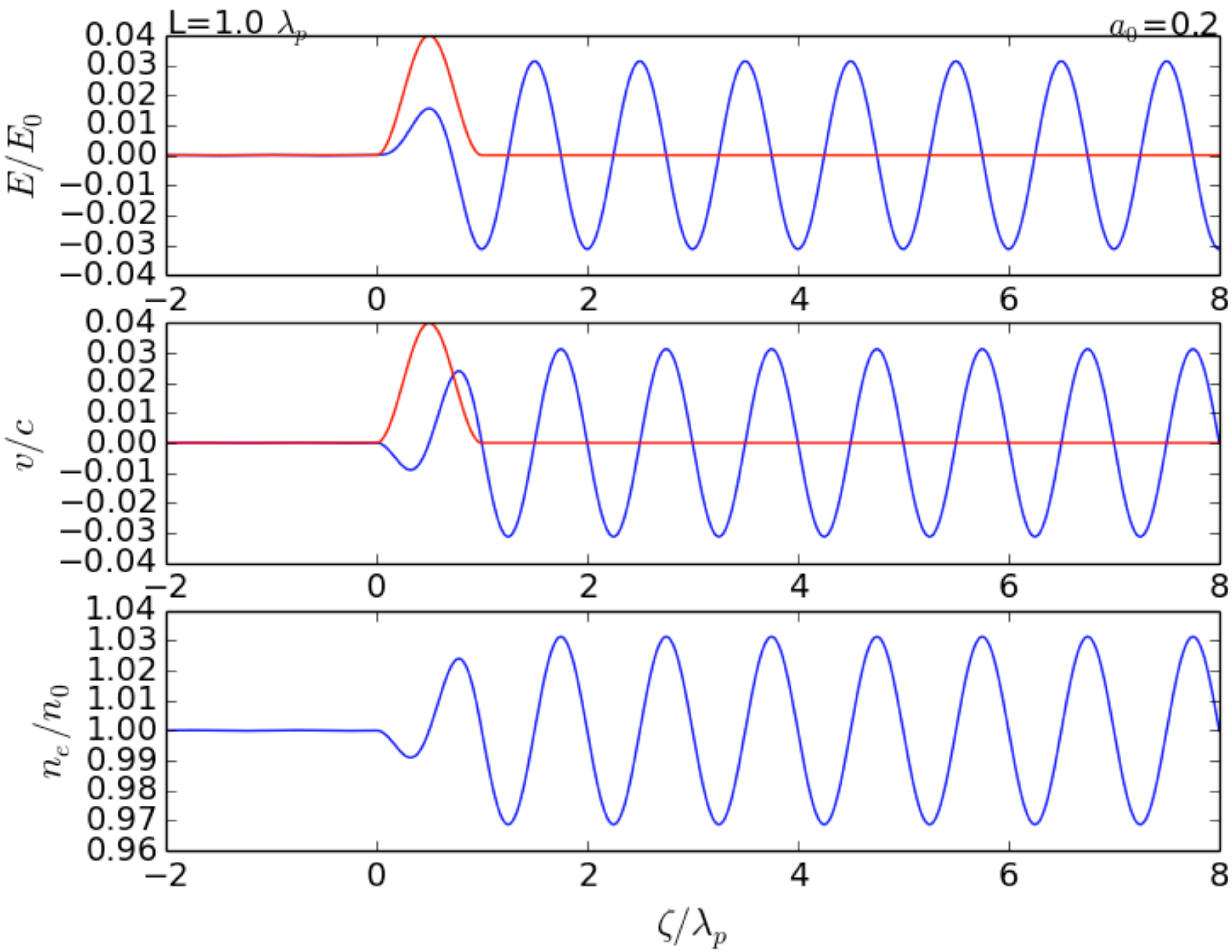
$$n_1 = n_0 \beta \qquad \text{(Continuity)}$$

$$\frac{\partial \beta}{\partial \zeta} = eE - \frac{\partial (a^2)}{\partial \zeta} \qquad \text{(Motion)}$$

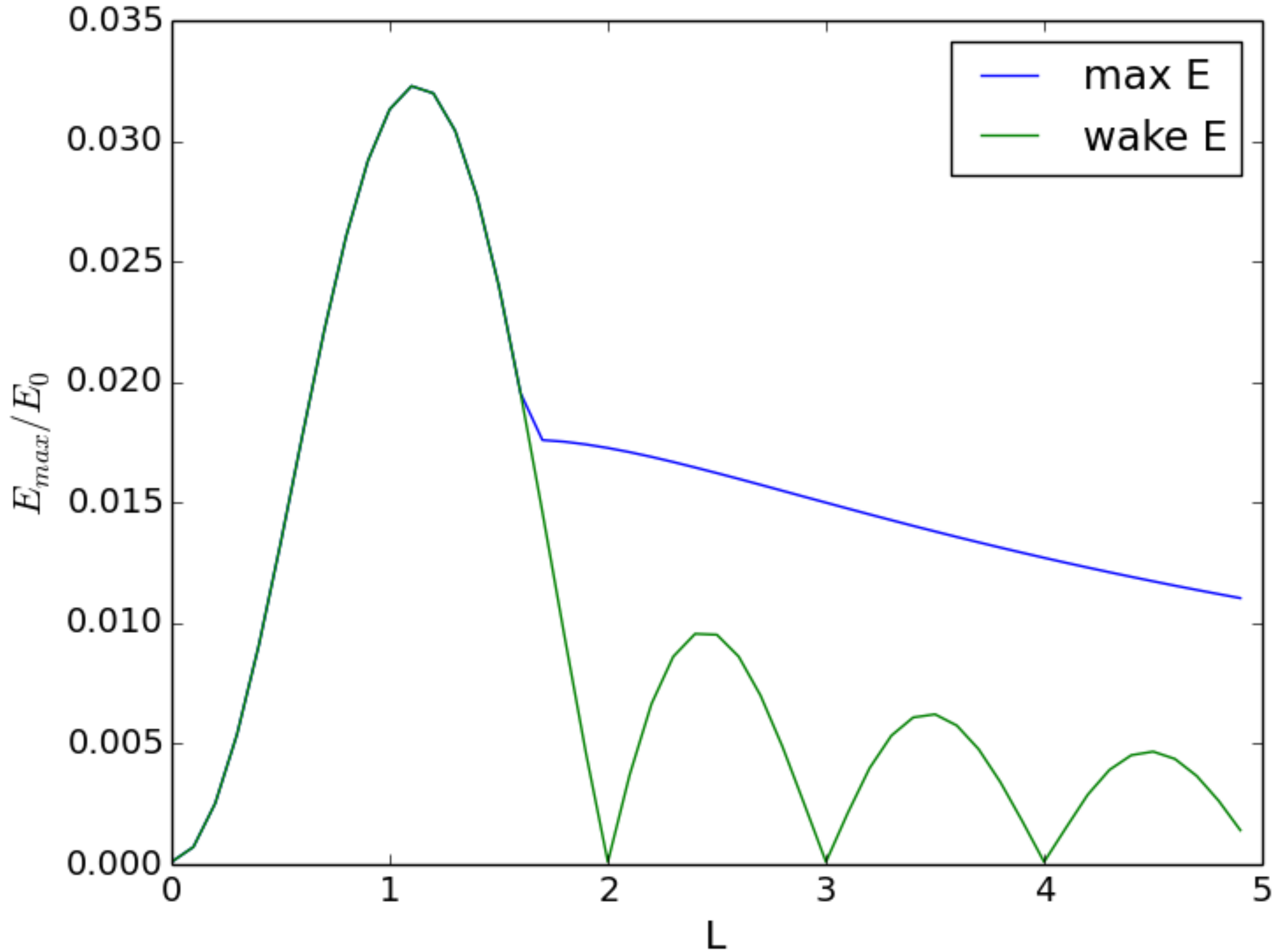
Assuming $\beta \ll 1$, $n_1 \ll n_0$ $\qquad$ $n_e = n_0(1+\beta)$

Have coupled equations in E and β to solve

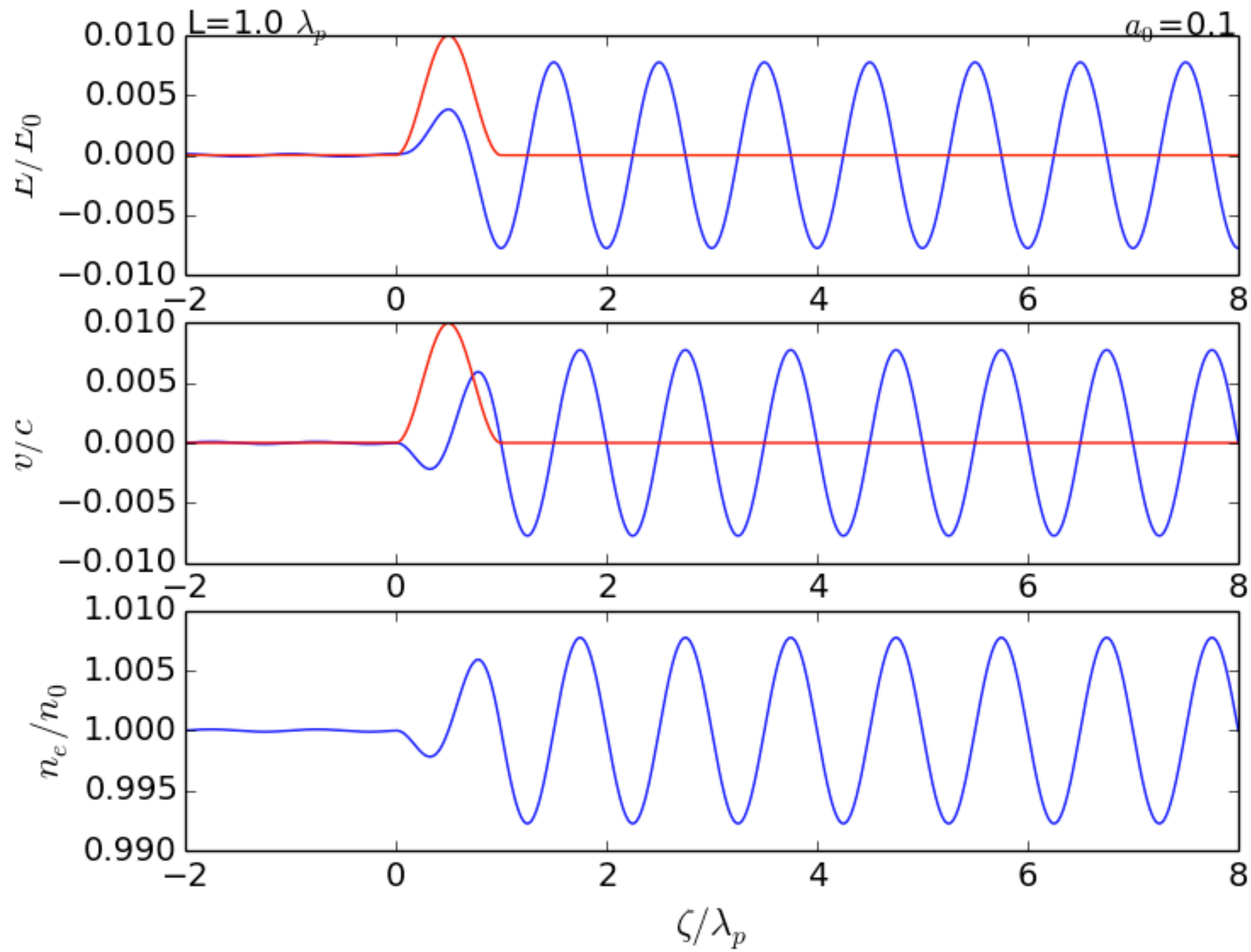
Wakefield generation



Wakefield generation



Wakefield generation



Wakefield generation

Now including plasma wave non-linearity

$$\frac{\partial E}{\partial \zeta} = -n_1 \qquad \text{(Gauss' Law)}$$

$$n_1 = (n_0 + n_1)\beta \qquad \text{(Continuity)}$$

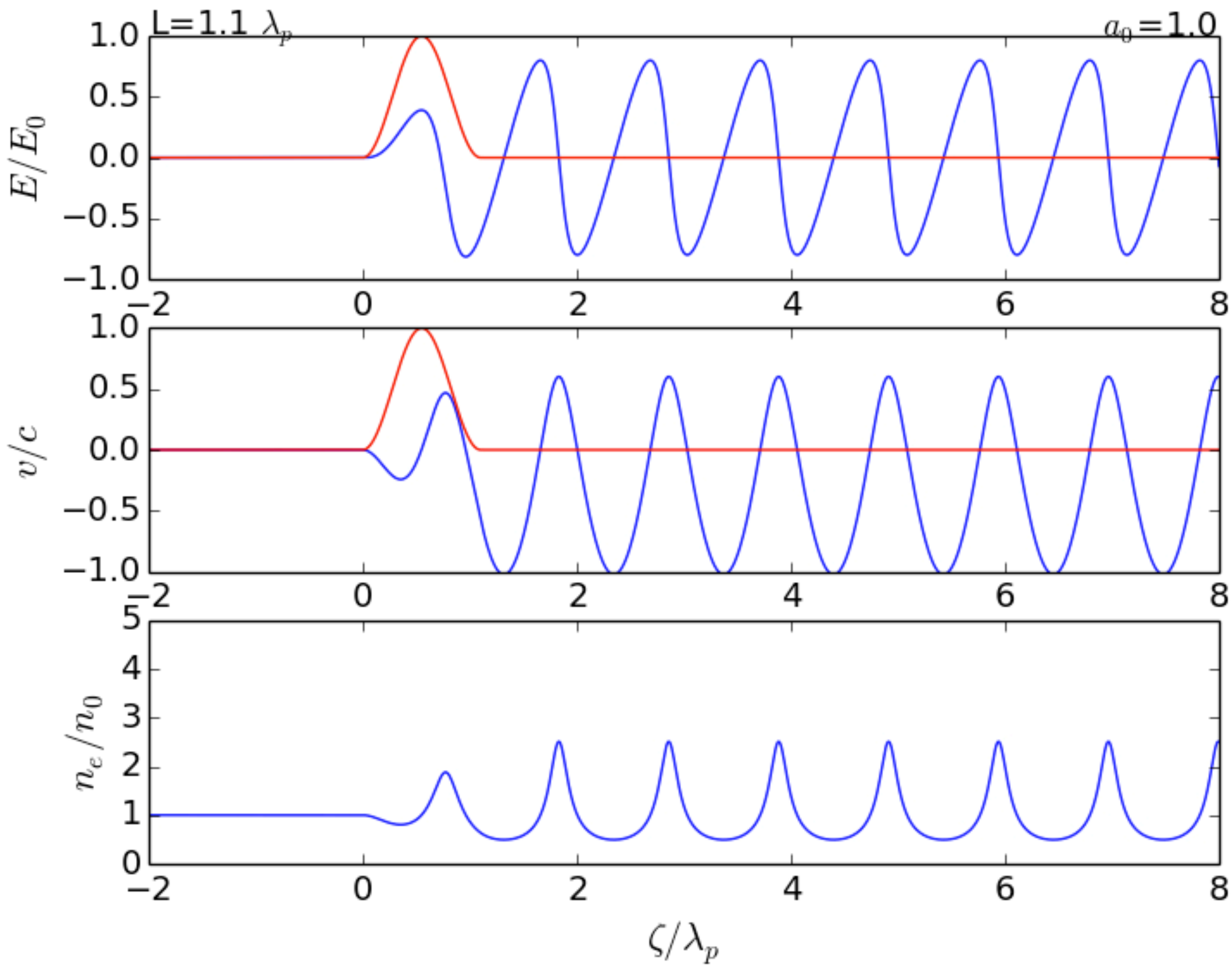
$$\frac{\partial \beta}{\partial \zeta} = eE - \frac{\partial (a^2)}{\partial \zeta} \qquad \text{(Motion)}$$

Rearranging Continuity equation:

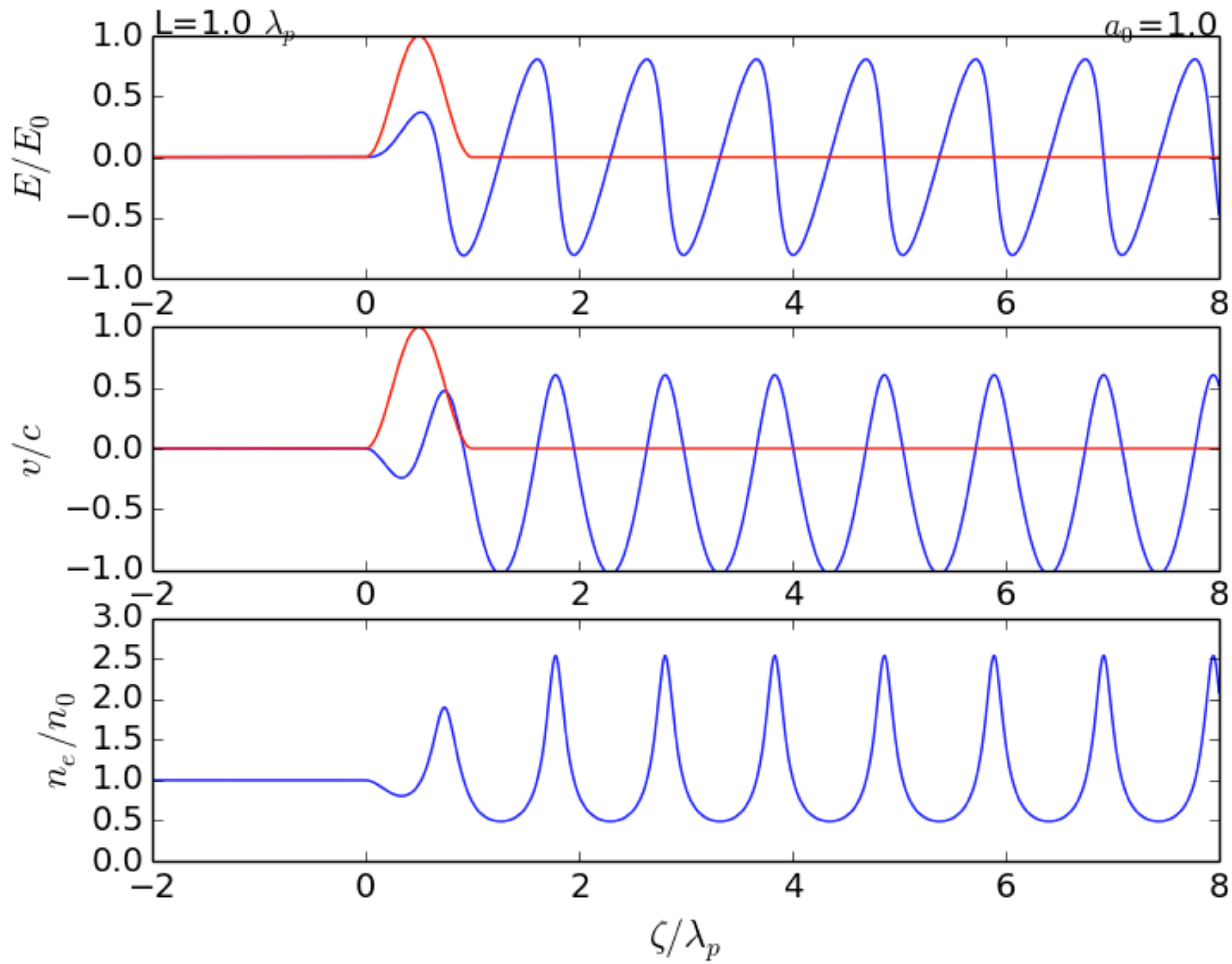
$$n_e = n_0 + n_1 = \frac{n_0}{1-\beta}$$

This implies that $n_{min} = \frac{1}{2}$ in 1D, though complete cavitation is possible in 3D.

Wakefield generation



Wakefield generation



Wakefield generation

Include relativity (and convection):

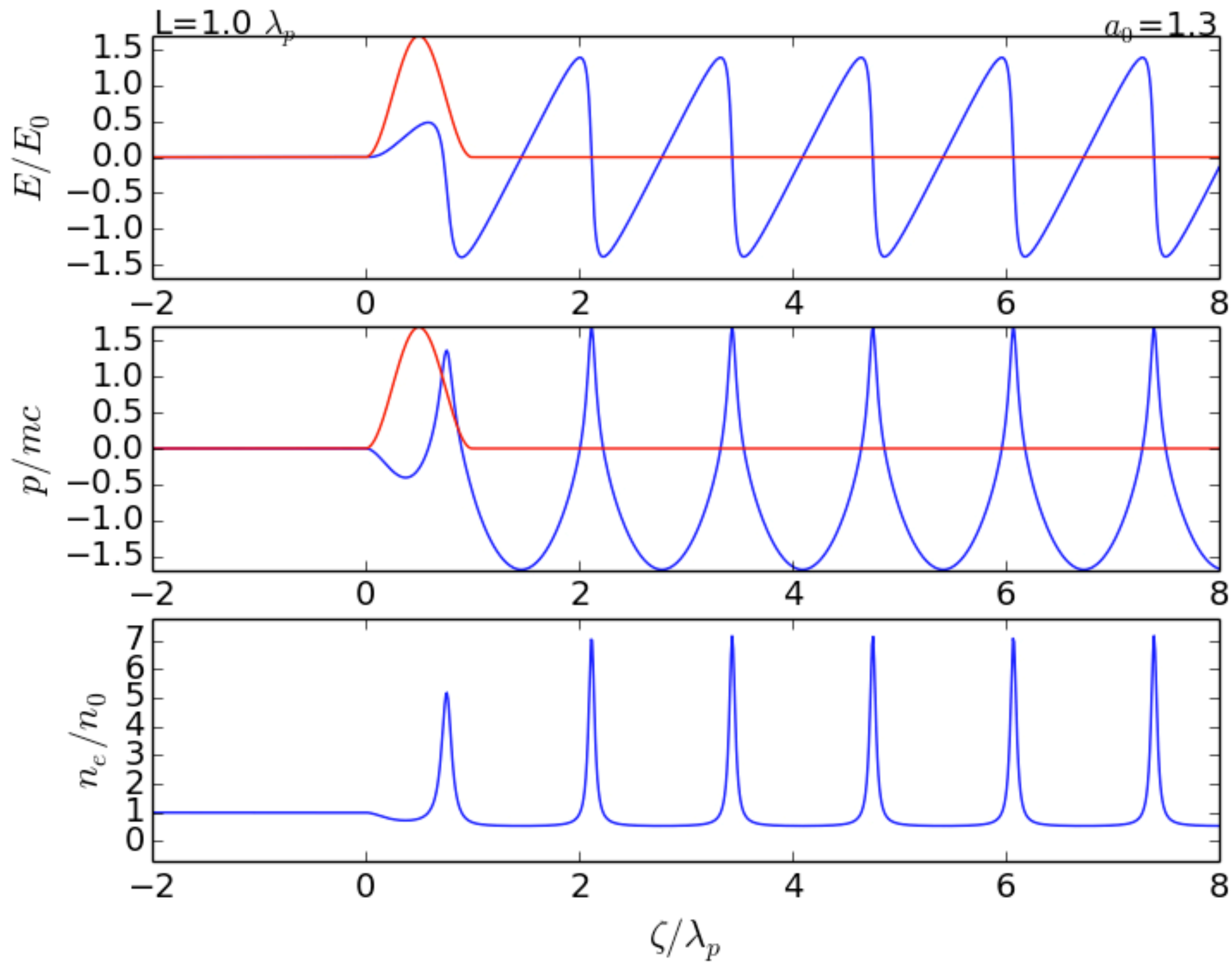
$$\frac{\partial E}{\partial \zeta} = -n_1 \qquad \text{(Gauss' Law)}$$

$$n_1 = (n_0 + n_1)\beta \qquad \text{(Continuity)}$$

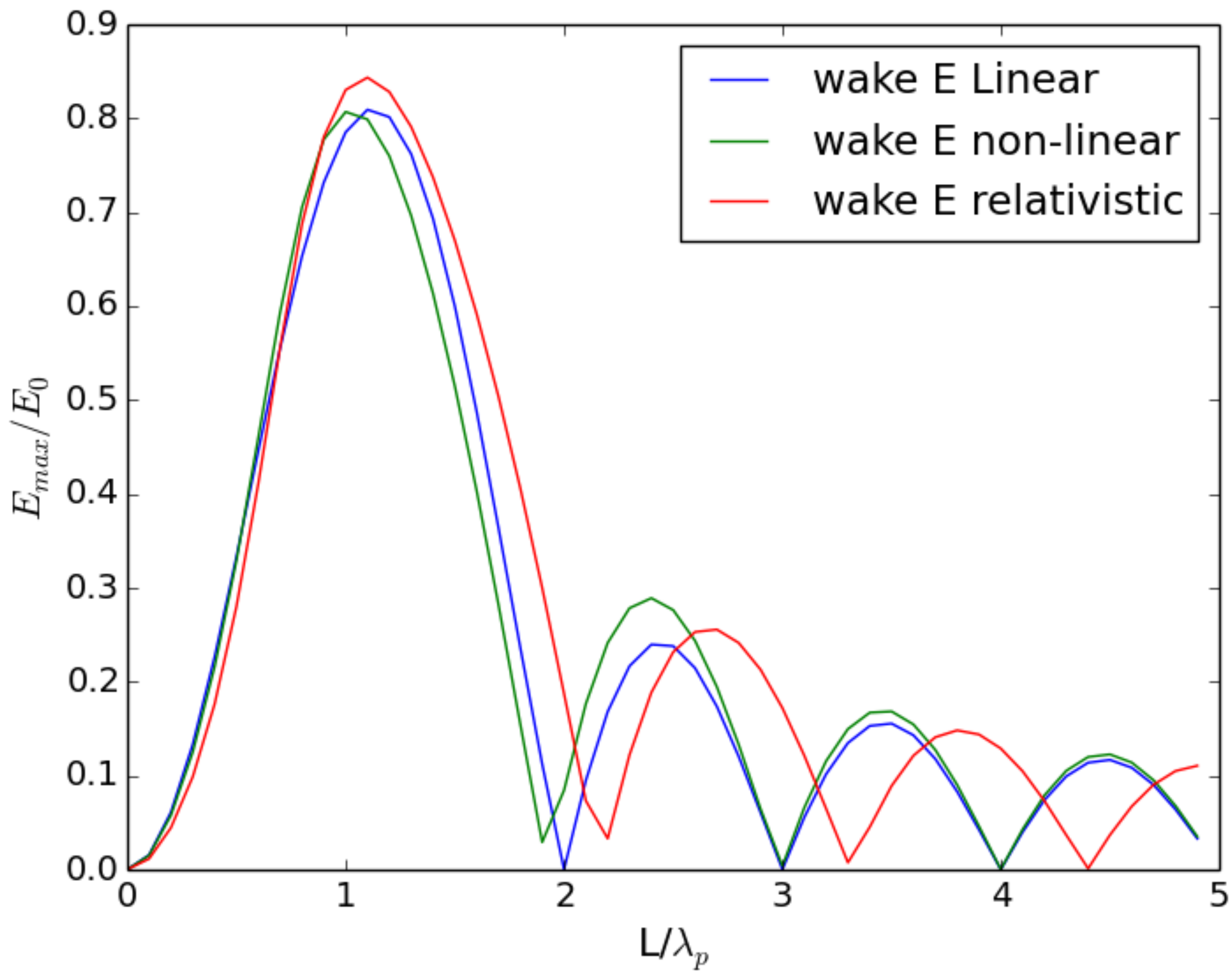
$$(1-\beta)\frac{\partial p}{\partial \zeta} = eE - \frac{1}{\gamma}\frac{\partial (a^2)}{\partial \zeta} \qquad \text{(Motion)}$$

$$\text{and using } \beta = \frac{p}{(1+{p_T}^2)^{1/2}} = \frac{p}{(1+p^2+a^2)^{1/2}}$$

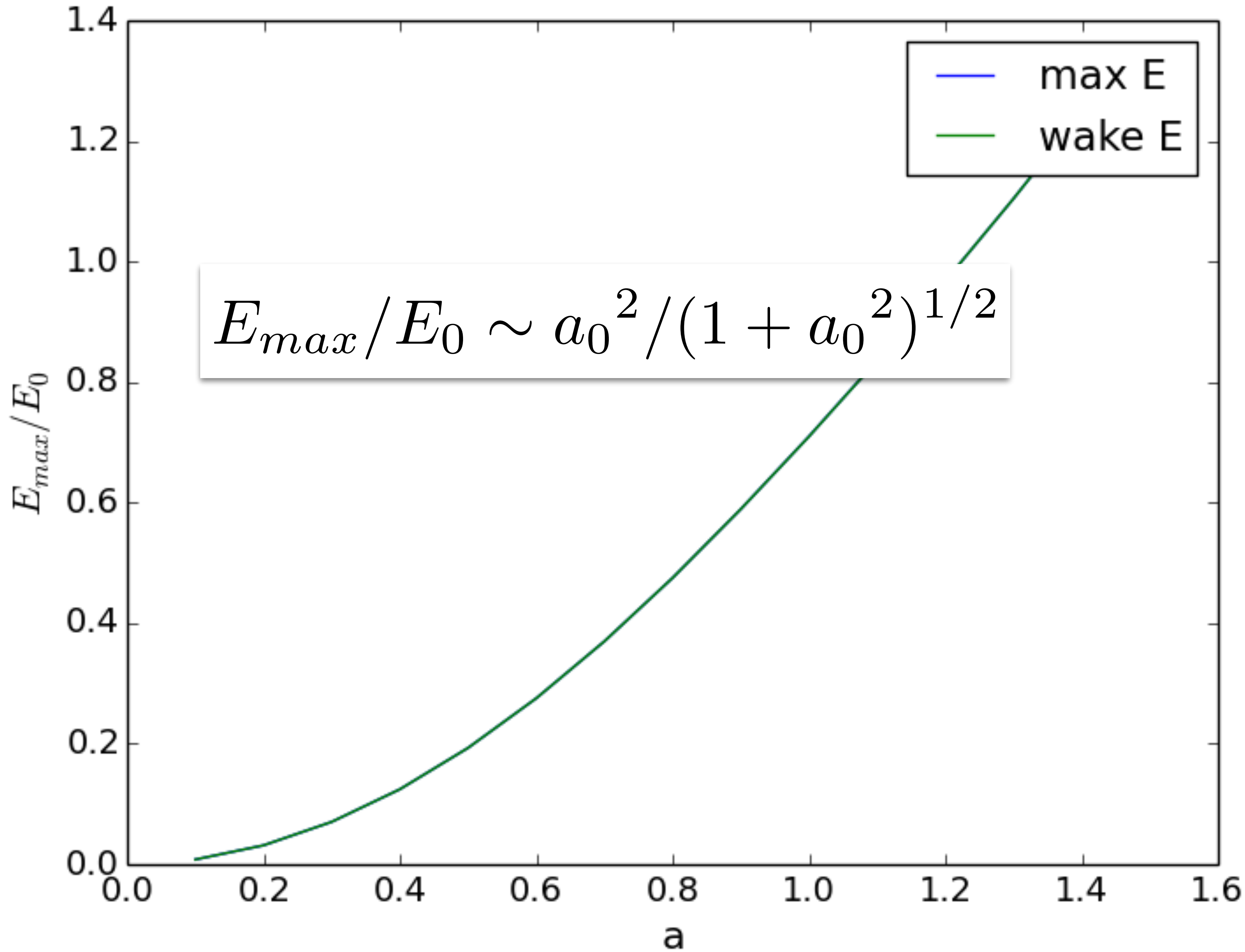
Wakefield generation



Wakefield generation



Wakefield generation



$$E_{max}/E_0 \sim {a_0}^2/(1+{a_0}^2)^{1/2}$$

Wakefield Generation Summary

Wake amplitude maximised for $L_{fwhm} \sim \lambda_p/2$

Continuity forces steepening of plasma waves, and sawtoothing of E-field

Non-linearities increases length over which plasma wave grows.

Relativity cause flattening of wakes and lengthening of plasma wave amplitude

$$E_{max}/E_0 \sim a_0^2/(1+a_0^2)^{1/2}$$

For linear wake of max amp. : $n_1 = n_0 \cos k_p \xi$

Gauss : $E = -\frac{e n_0}{\epsilon_0 k_p} \sin k_p \xi$

(potential) : $\phi = -\frac{e n_0}{\epsilon_0 {k_p}^2} \cos k_p \xi$

Max energy gain? : W $= -2e[\phi_{min} - \phi max] = 2 \frac{n_0 e^2}{\epsilon_0 {k_p}^2}$

$$= 2 \frac{n_0 e^2}{\epsilon_0 m} \frac{mc^2}{{\omega_p}^2} = 2mc^2 \approx 1\,\mathrm{MeV}$$

For linear wake of max amp. : $$n_1 = n_0 \cos k_p \xi$$

Gauss : $$E = -\frac{e n_0}{\epsilon_0 k_p} \sin k_p \xi$$

(potential) : $$\phi = -\frac{e n_0}{\epsilon_0 {k_p}^2} \cos k_p \xi$$

Consider boosted frame : $E' = E, \quad \text{but } k_p \to {k_p}' = k_p/\gamma$

So : $$E' = -\frac{e n_0}{\epsilon_0 k_p} \sin\left({k_p}' \xi' / \gamma\right)$$

potential : $$\phi' = -\gamma \frac{e n_0}{\epsilon_0 {k_p}^2} \cos\left({k_p}' \xi' / \gamma\right)$$

Energy gain in boosted frame : $W = 2\gamma mc^2$

BUT electrons focussed over half of length :

Energy gain in boosted frame : $W = \gamma mc^2$

Then : $p \simeq \gamma mc$

Back to lab frame : $\begin{pmatrix} iW \\ cp \end{pmatrix} = \begin{pmatrix} \gamma & i\beta\gamma \\ -i\beta\gamma & p \end{pmatrix} \begin{pmatrix} iW' \\ cp \end{pmatrix}$

$iW = \gamma W' + i\beta\gamma cp$

$W \simeq \gamma^2 mc^2 + \gamma^2 mc^2 = 2\gamma^2 mc^2$

Acceleration length in boosted frame : $L' = 2\pi/{k_p}' = \gamma 2\pi/k_p$

Lorentz transforming :

$$v_\phi = v_g = \frac{\partial \omega}{\partial k} = \eta c = c\sqrt{1 - {\omega_p}^2/\omega^2} \qquad \rightarrow \qquad \beta_\phi = \eta$$

$$\gamma_\phi = (1 - {\beta_\phi}^2)^{-1/2} = (1 - (1 - {\omega_p}^2/\omega^2)^{-1/2}$$

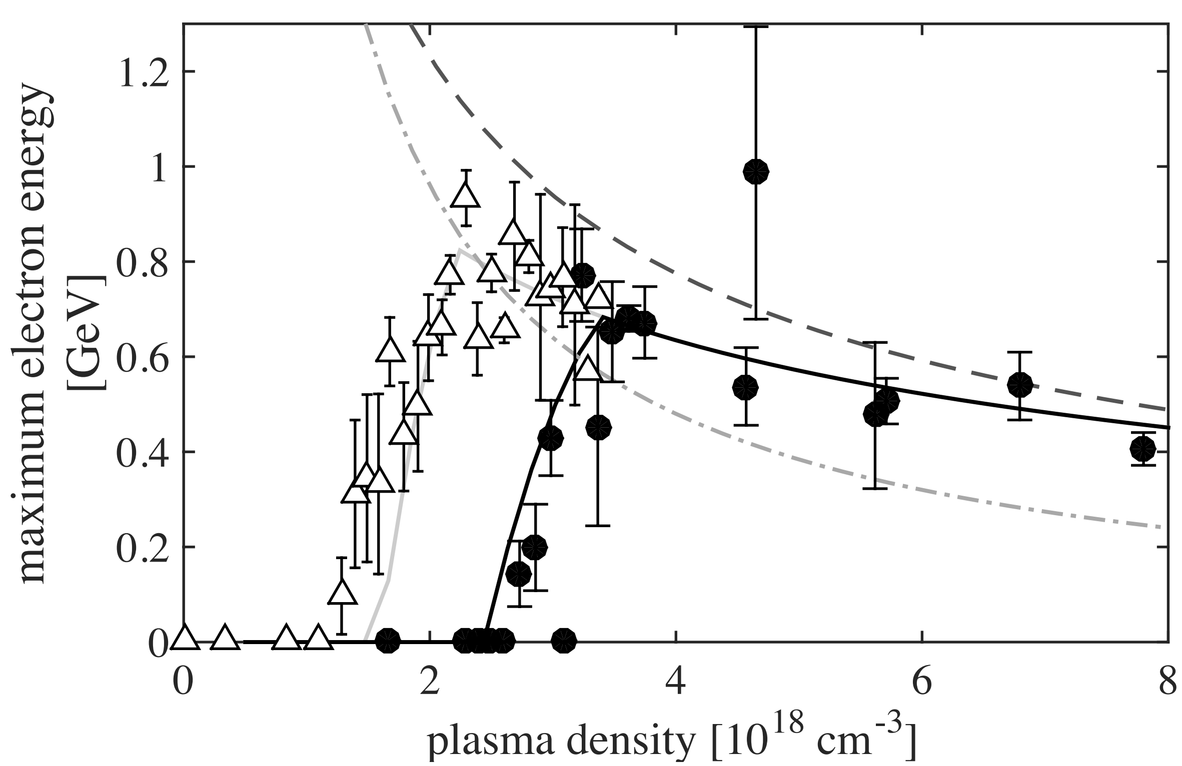
$$\gamma_\phi = \frac{\omega_0}{\omega_p} = \left(\frac{n_{cr}}{n_0}\right)$$

So : $W = 2mc^2 \cdot \dfrac{n_{cr}}{n_0}$

$$\text{Acceleration length in boosted frame}: L' = 2\pi/k_p{}' = \gamma 2\pi/k_p$$

$$\text{Lorentz transforming}: L_{deph} = \gamma^2 2\pi/k_p = \gamma^2 2\pi c/\omega_p$$

$$L_{deph} = \frac{n_{cr}}{n_0}\frac{\omega_0}{\omega_p}\frac{2\pi c}{\omega_0} = \left(\frac{n_{cr}}{n_0}\right)^{3/2} \lambda_0$$



M. Bloom, Phd Thesis (Imperial College London, (2012)

Formula Summary

Regime	a_0	$k_p w_0$	$\delta n/n_0$	$k_p L_{deph}$	$k_p L_{depl}$	λ_W	γ_ϕ	$\Delta W/mc^2$
Linear:	<1	2π	${a_0}^2$	$\frac{{\omega_0}^2}{{\omega_p}^2}$	$\left(\frac{{\omega_0}^2}{{\omega_p}^2}\right)\left(\frac{\omega_p\tau}{{a_0}^2}\right)$	$\frac{2\pi}{k_p}$	$\frac{\omega_0}{\omega_p}$	${a_0}^2\left(\frac{{\omega_0}^2}{{\omega_p}^2}\right)$
1D NL:	>1	2π	a_0	$4{a_0}^2\left(\frac{{\omega_0}^2}{{\omega_p}^2}\right)$	$\frac{1}{3}\left(\frac{{\omega_0}^2}{{\omega_p}^2}\right)\omega_p\tau$	$\frac{4a_0}{k_p}$	$\sqrt{a_0}\left(\frac{\omega_0}{\omega_p}\right)$	$4{a_0}^2\left(\frac{{\omega_0}^2}{{\omega_p}^2}\right)$
3D NL:	>2	$2\sqrt{a_0}$	$\frac{1}{2}\sqrt{a_0}$	$\frac{4}{3}\sqrt{a_0}\left(\frac{{\omega_0}^2}{{\omega_p}^2}\right)$	$\left(\frac{{\omega_0}^2}{{\omega_p}^2}\right)\omega_p\tau$	$\frac{2\pi\sqrt{a_0}}{k_p}$	$\frac{1}{\sqrt{3}}\left(\frac{\omega_0}{\omega_p}\right)$	$\frac{2}{3}{a_0}^2\left(\frac{{\omega_0}^2}{{\omega_p}^2}\right)$
Bubble:	>20	$\sqrt{a_0}$	$\sqrt{a_0}$		$a_0\left(\frac{{\omega_0}^2}{{\omega_p}^2}\right)\omega_p\tau$			$4{a_0}^2\left(\frac{{\omega_0}^2}{{\omega_p}^2}\right)$

W. Lu et al, PR STAB 10, 061301 (2007)