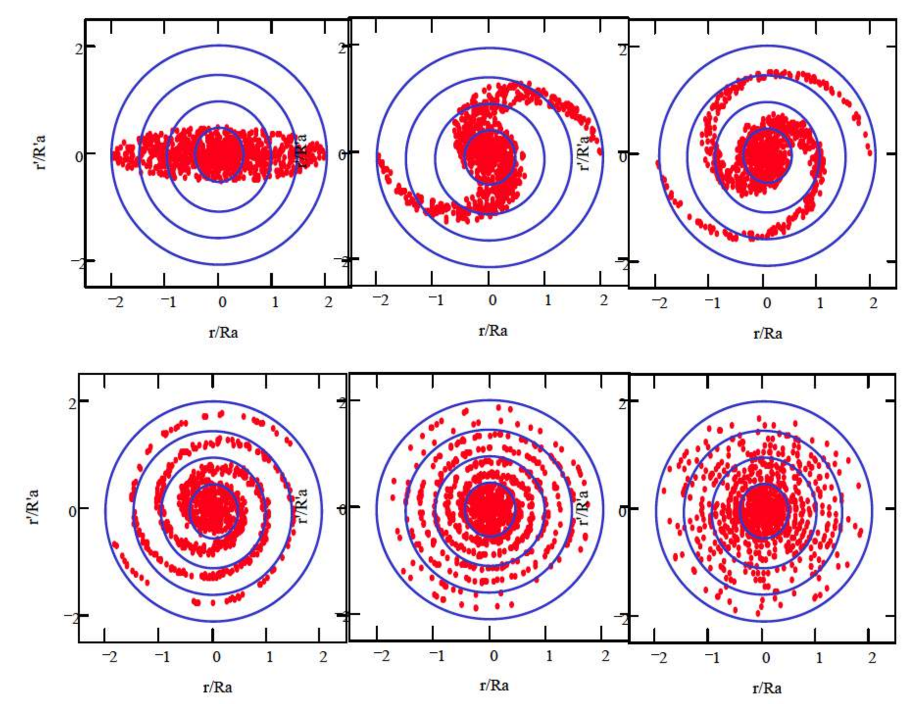
Electron beam properties and FEL – Lecture III

Massimo.Ferrario@LNF.INFN.IT



EuPRAXIA

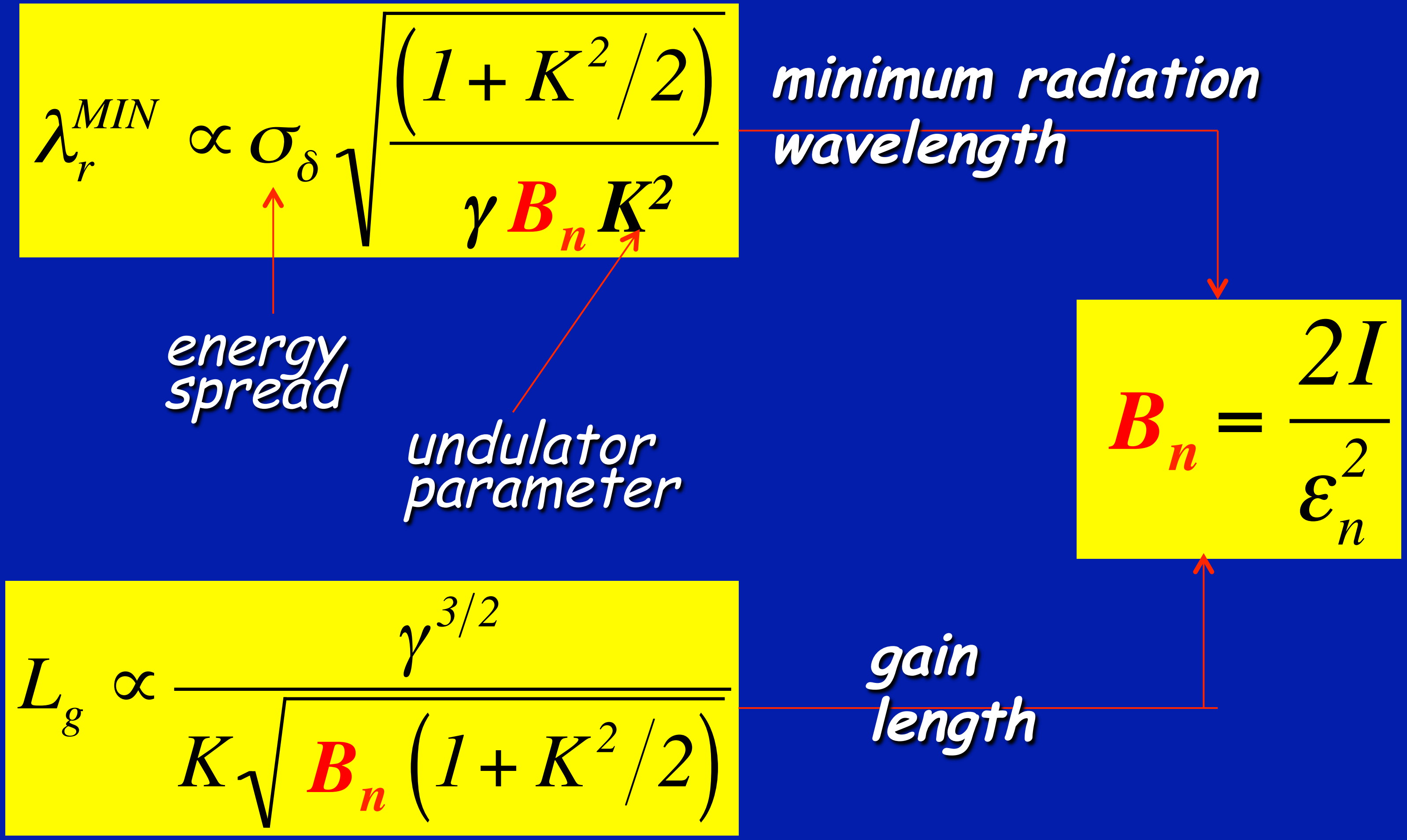
Advanced Summer School on

"Laser-Driven Sources of High Energy Particles and Radiation"

9-16 July 2017, CNR Conference Centre, Anacapri, Capri, Italy

SPARC LAB

SASE FEL Electron Beam Requirements: High Brightness B_n



$$\lambda_r^{MIN} \propto \sigma_\delta \sqrt{\frac{\left(1+K^2/2\right)}{\gamma B_n K^2}}$$

minimum radiation wavelength

energy spread

undulator parameter

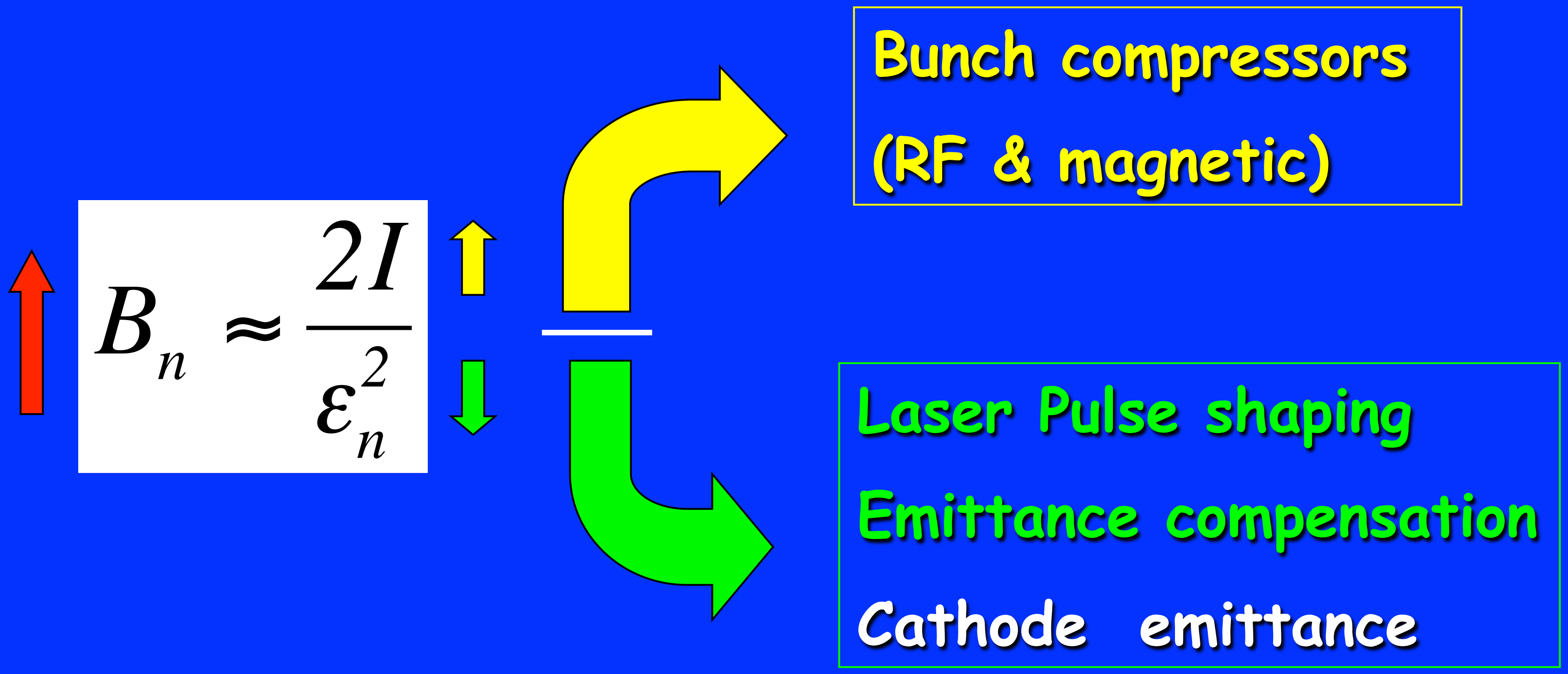
$$B_n = \frac{2I}{\varepsilon_n^2}$$

$$L_g \propto \frac{\gamma^{3/2}}{K\sqrt{B_n\left(1+K^2/2\right)}}$$

gain length

R. Saldin et al. in *Conceptual Design of a 500 GeV e+e- Linear Collider with Integrated X-ray Laser Facility*, DESY-1997-048

Short Wavelength SASE FEL Electron Beam Requirement: High Brightness $B_n > 10^{15}$ A/m^2



$$\uparrow B_n \approx \frac{2I}{\varepsilon_n^2}\;\begin{matrix}\uparrow\\ \downarrow\end{matrix}$$

Bunch compressors

(RF & magnetic)

Laser Pulse shaping

Emittance compensation

Cathode emittance

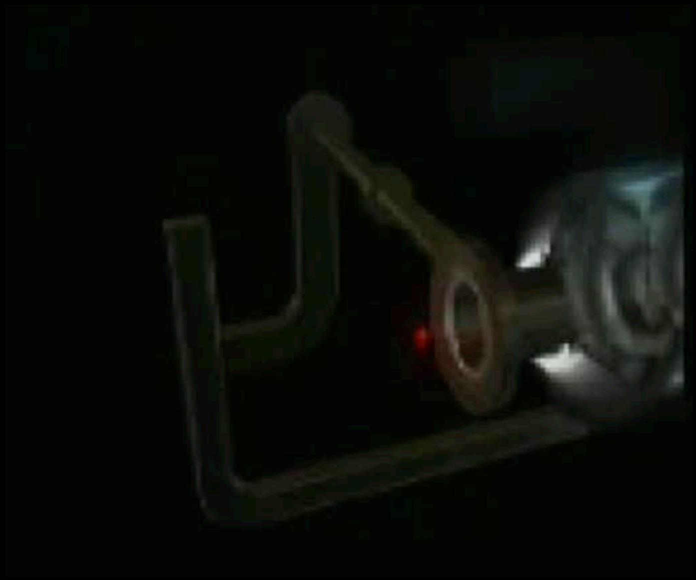
Longitudinal Manipulation

The problem of relativistic bunch length

Low energy electron bunch injected in a linac:

$$\gamma \approx 1$$

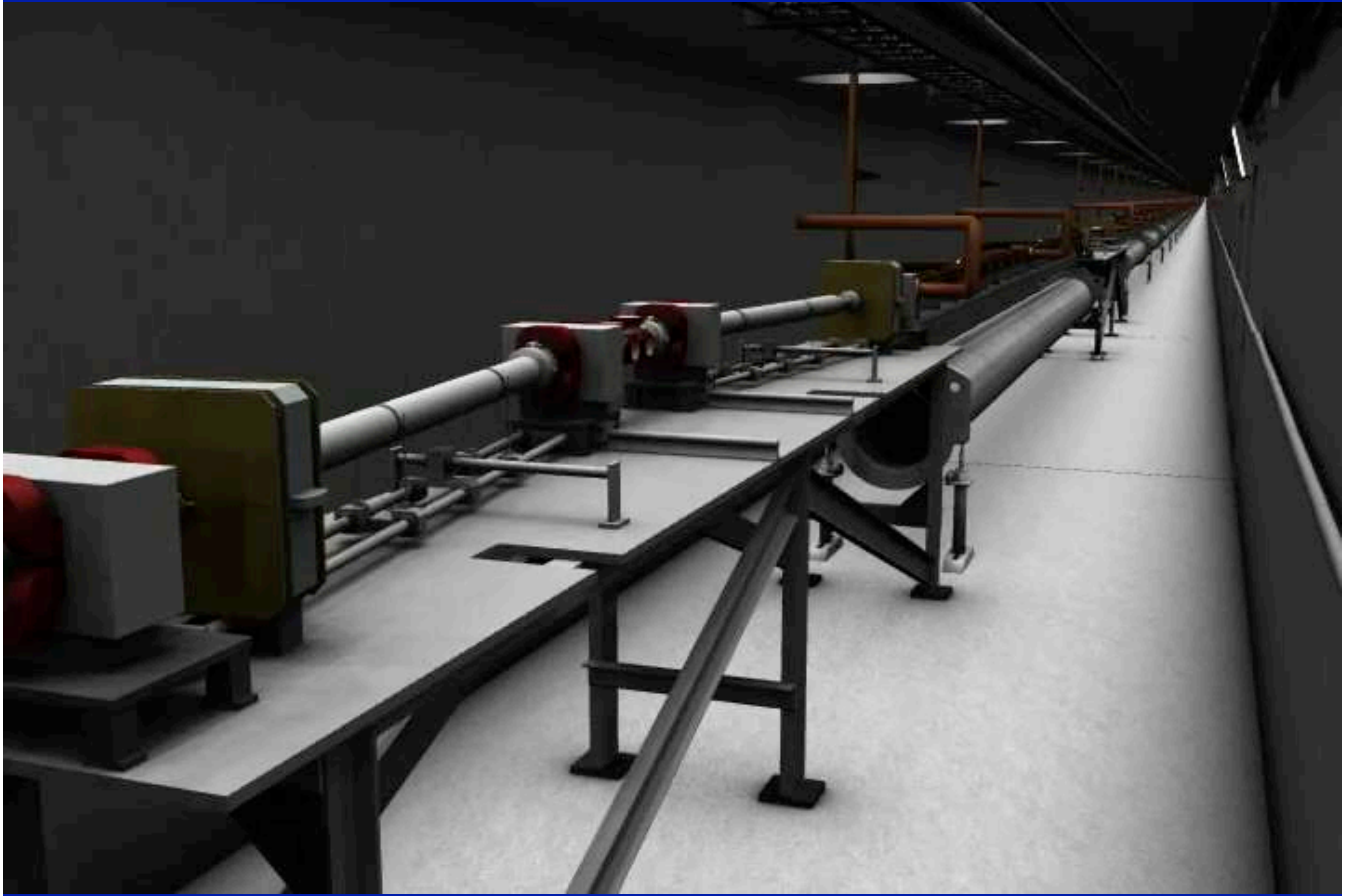
$$L_b = 3mm \approx L_b'$$



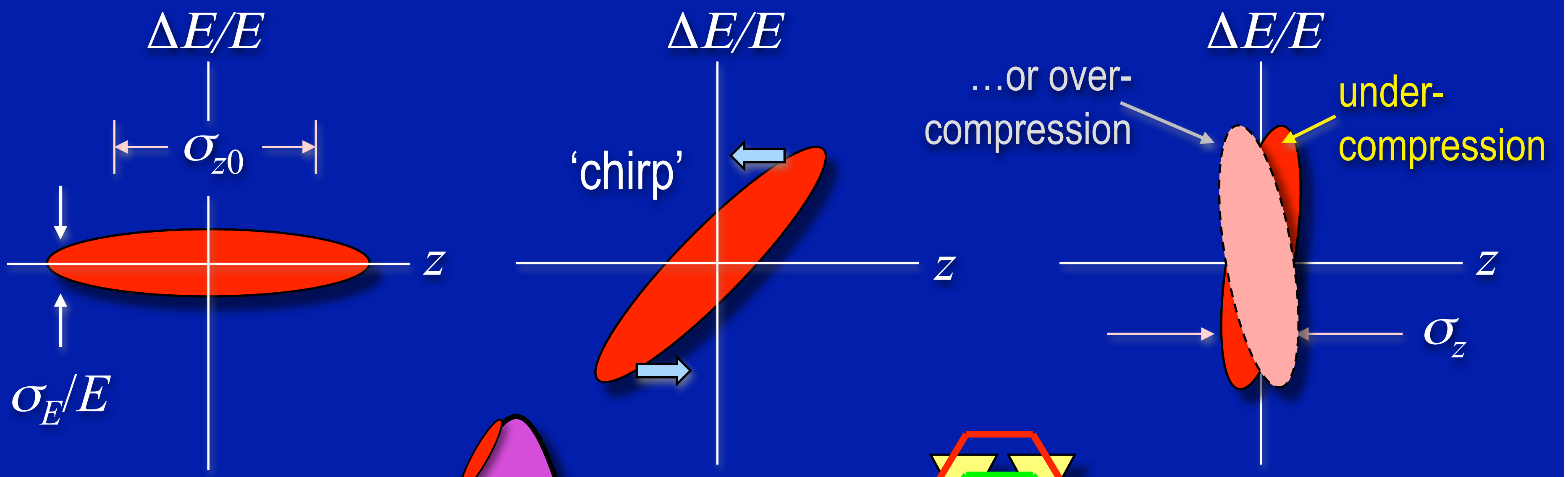
Length contraction?

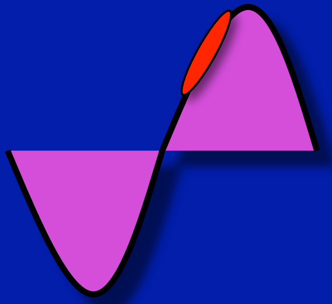
$$\xcancel{\begin{gathered}\gamma = 1000 \\ L_b = \frac{L_b'}{\gamma} = 3\mu m\end{gathered}}$$

Magnetic compressor (Chicane)



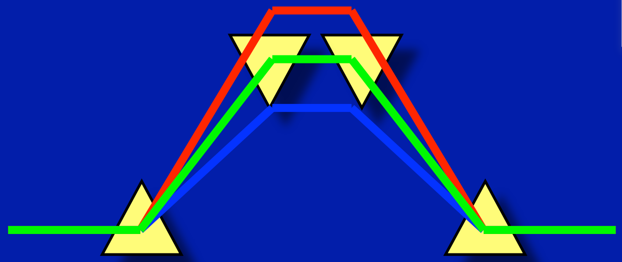
Magnetic compressor (Chicane)





$$V = V_0 \sin(\omega\tau)$$

RF Accelerating Voltage



$$\Delta z = R_{56}\Delta E/E$$

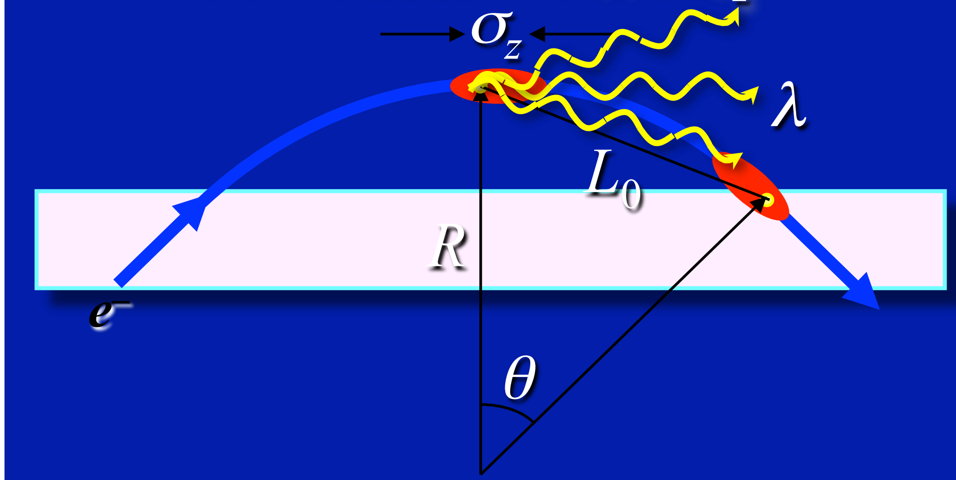
Path Length-Energy Dependent Beamline

Courtesy
P. Emma

Coherent Synchrotron Radiation (CSR)

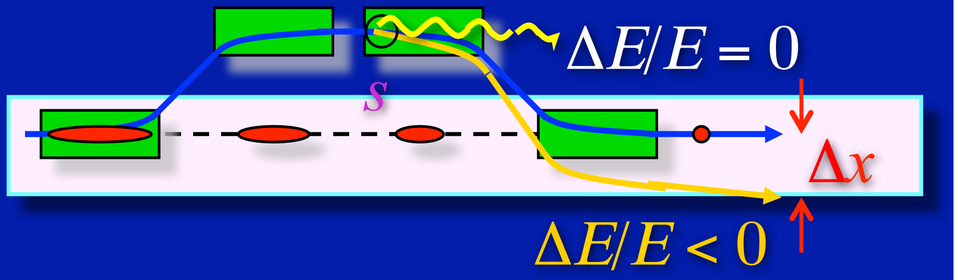
- Powerful radiation generates energy spread in bends
- Energy spread breaks achromatic system
- Causes bend-plane emittance growth (short bunch worse)

coherent radiation for $\lambda > \sigma_z$



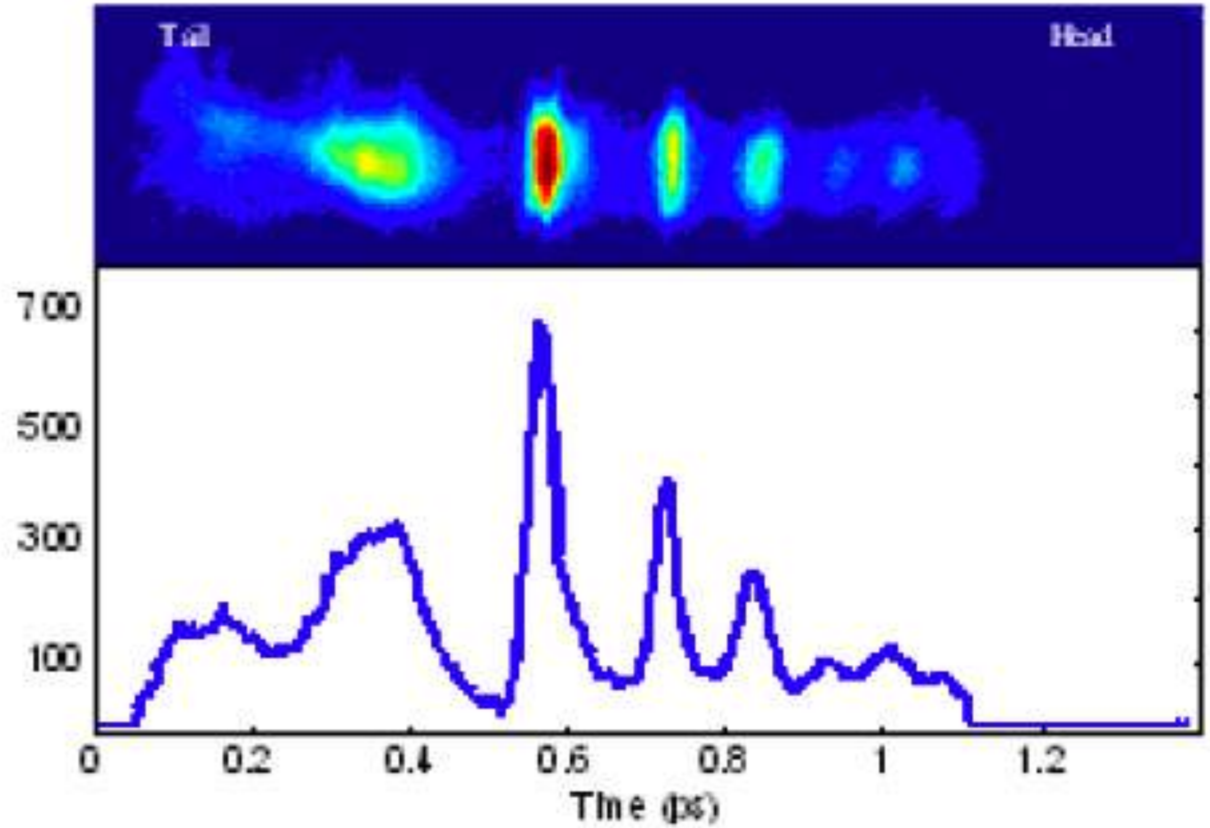
overtaking length: L0 ≈ $(24\sigma_z R^2)^{1/3}$

bend-plane emittance growth



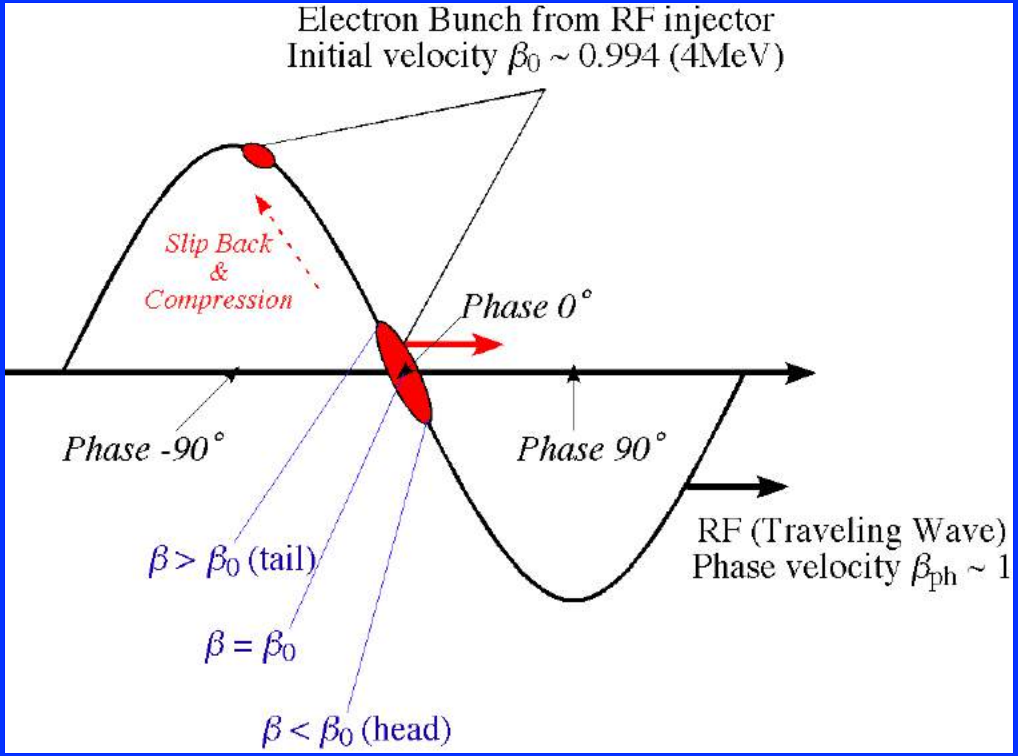
$\Delta x = R_{16}(s)\Delta E/E$

Courtesy
P. Emma


Tail
Head
700
500
300
100
0
0.2
0.4
0.6
0.8
1
1.2
Time (ps)

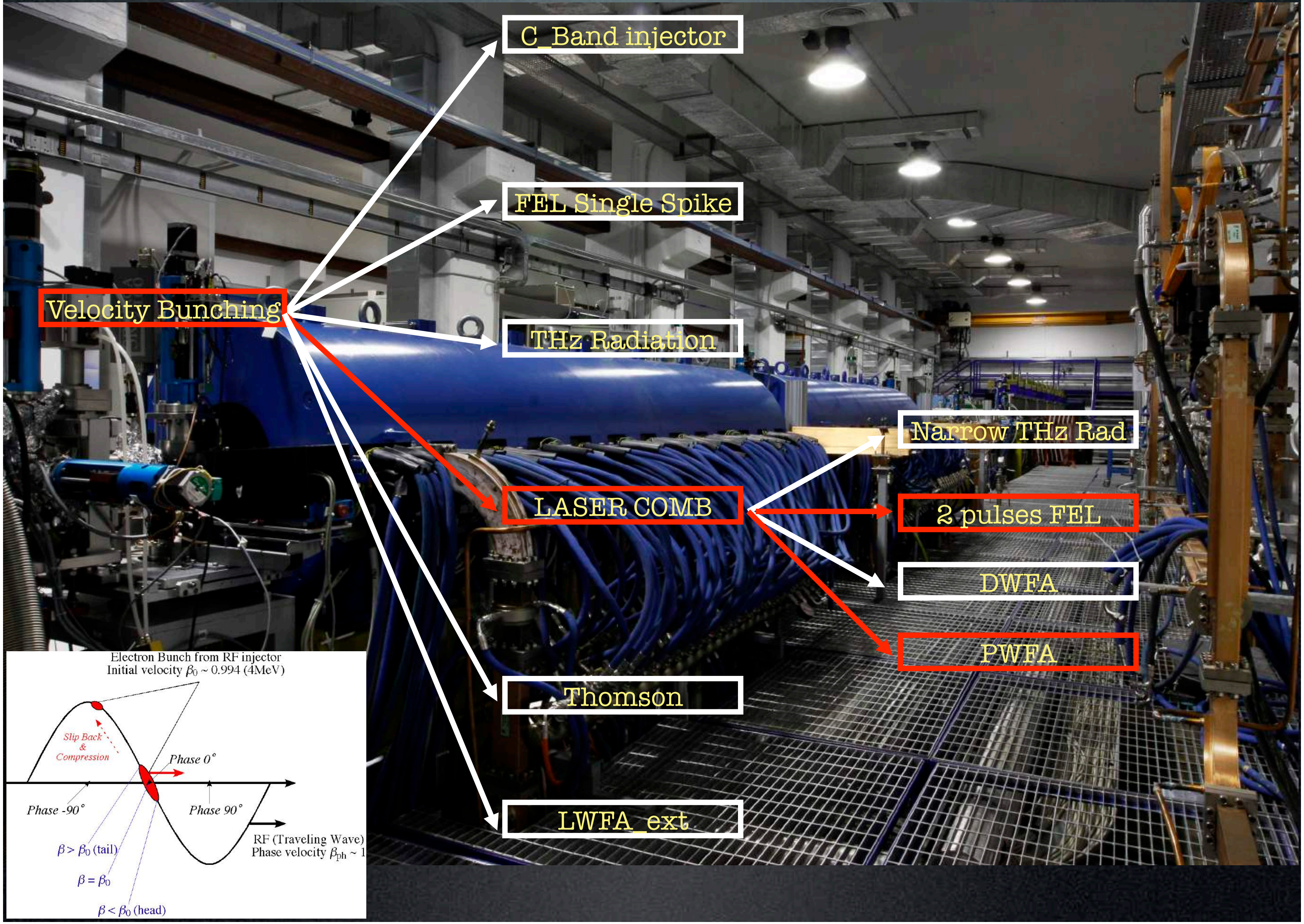
Velocity bunching concept (RF Compressor)

If the beam injected in a long accelerating structure at the crossing field phase and it is slightly slower than the phase velocity of the RF wave , it will slip back to phases where the field is accelerating, but at the same time it will be chirped and compressed.



The key point is that compression and acceleration take place at the same time within the same linac section, actually the first section following the gun, that typically accelerates the beam, under these conditions, from a few MeV (> 4) up to 25-35 MeV.

- **L. Serafini and M. Ferrario,** *Velocity Bunching in Photo-injectors,* Physics of, and Science with the X-Ray Free-Electron Laser, ed..by S. Chattopadhyay et al. © 2001 American Institute of Physics
- **M. Ferrario et al.,** *Experimental Demonstration of Emittance Compensation with Velocity Bunching,* Phys. Rev. Lett. **104**, 054801 (2010)


Velocity Bunching

C_Band injector

FEL Single Spike

THz Radiation

LASER COMB

Thomson

LWFA_ext

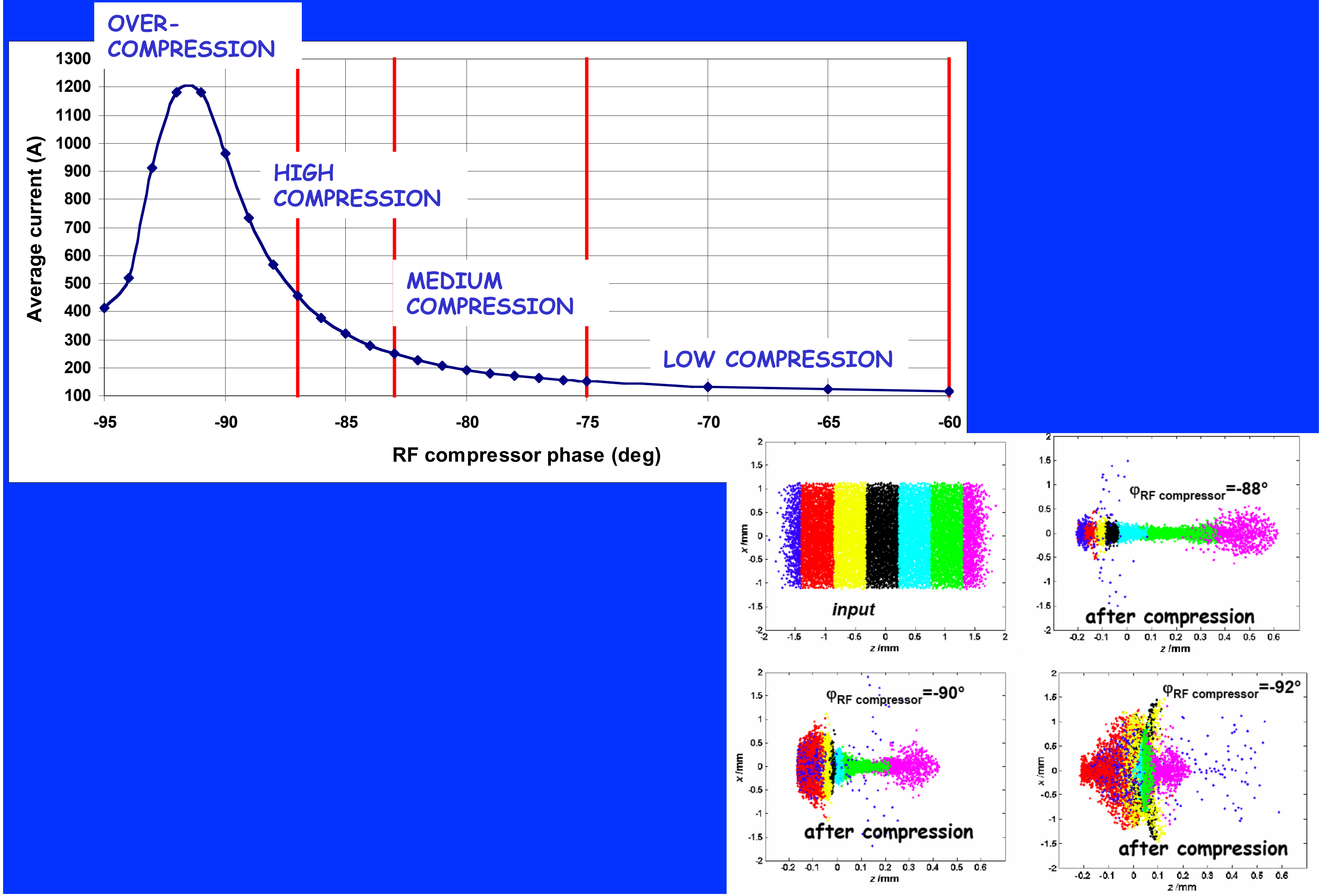
Narrow THz Rad

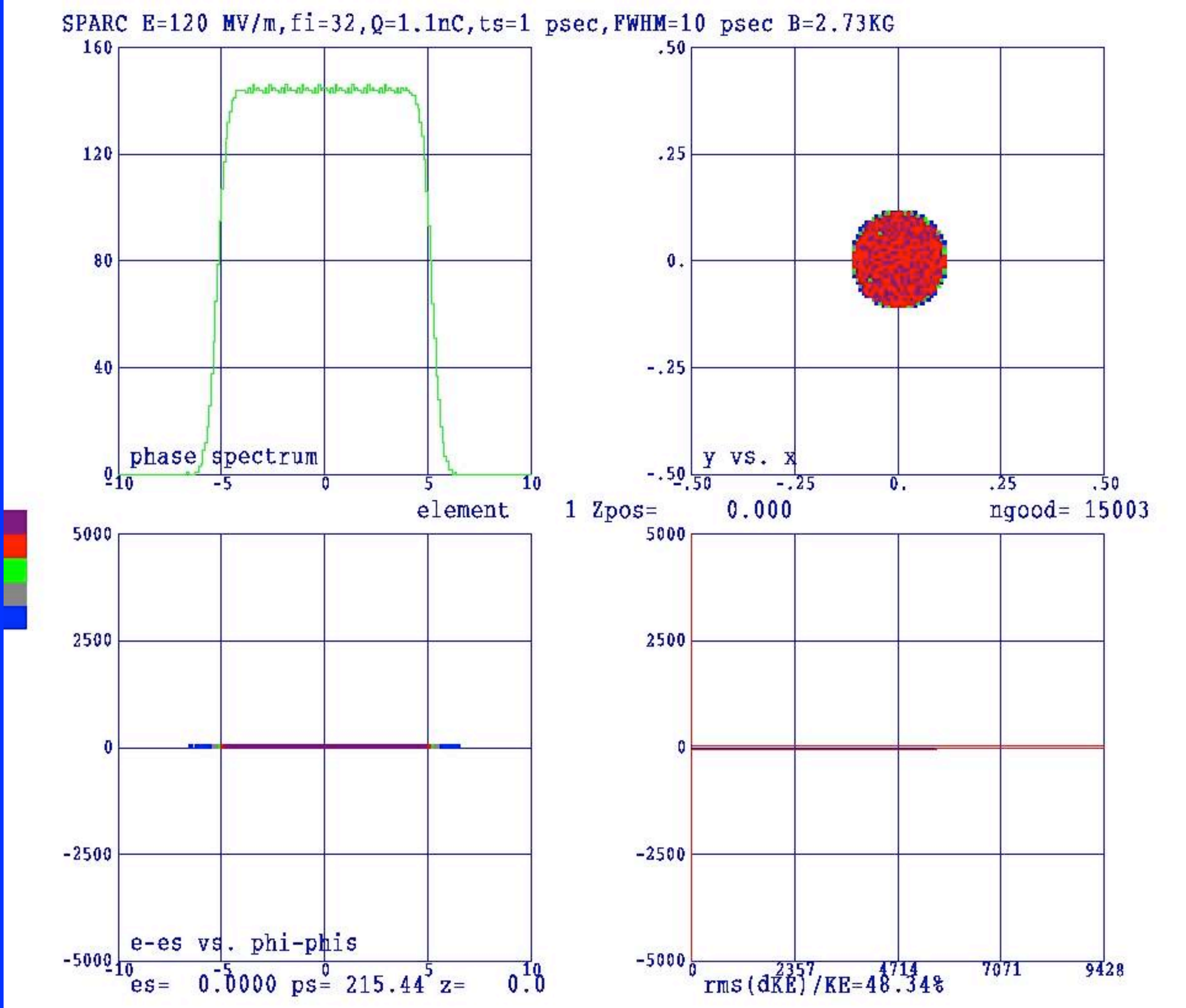
2 pulses FEL

DWFA

PWFA

Average current vs RF compressor phase





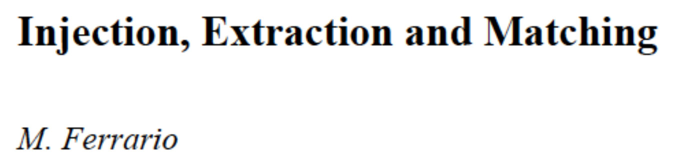
SPARC E=120 MV/m,fi=32,Q=1.1nC,ts=1 psec,FWHM=10 psec B=2.73KG

element 1 Zpos= 0.000 ngood= 15003

es= 0.0000 ps= 215.44 z= 0.0

rms(dKE)/KE=48.34%

Transverse Beam Dynamics

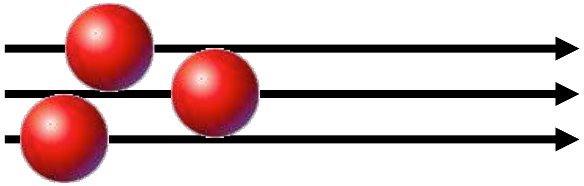


Injection, Extraction and Matching

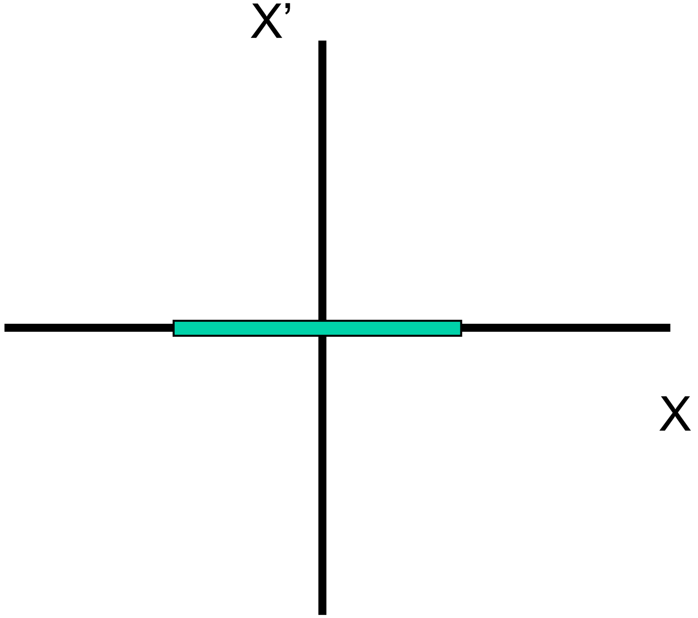
M. Ferrario

https://arxiv.org/ftp/arxiv/papers/1705/1705.10564.pdf

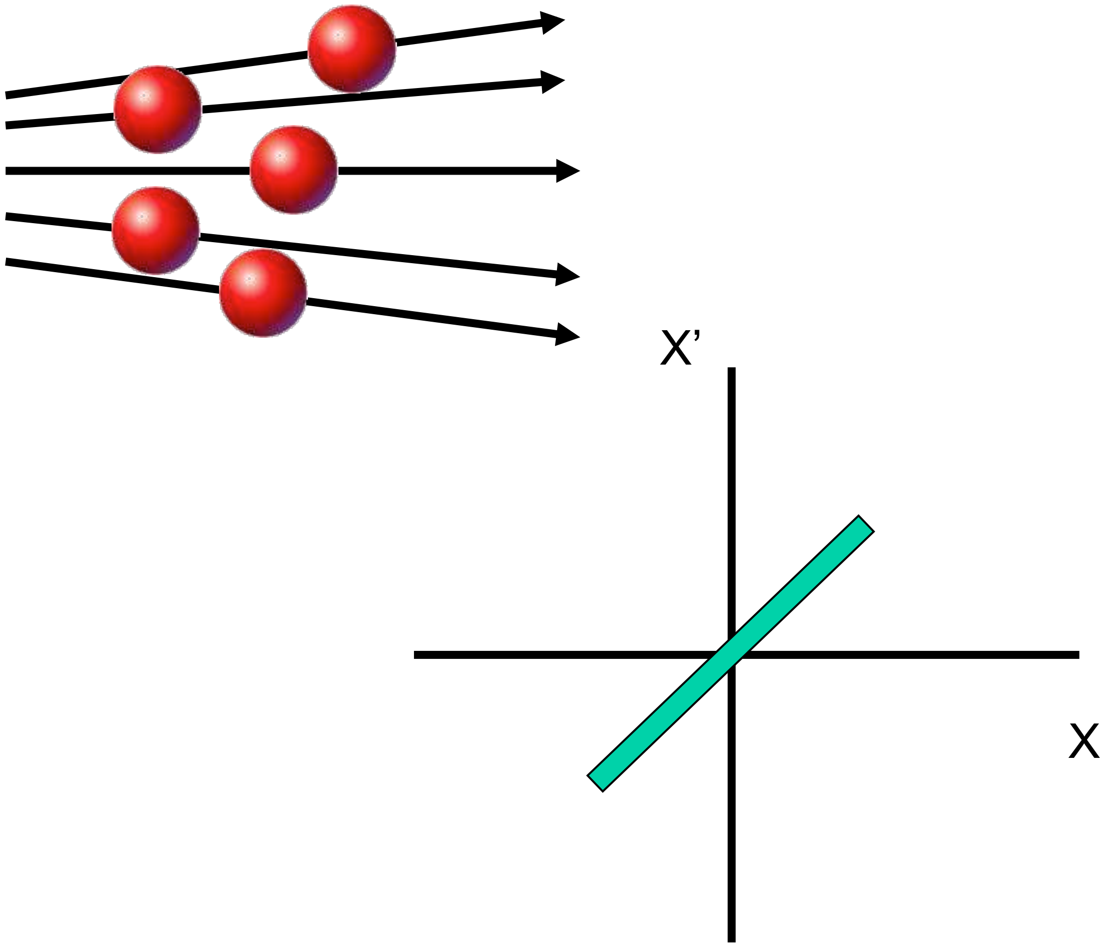
Trace space of an ideal laminar beam



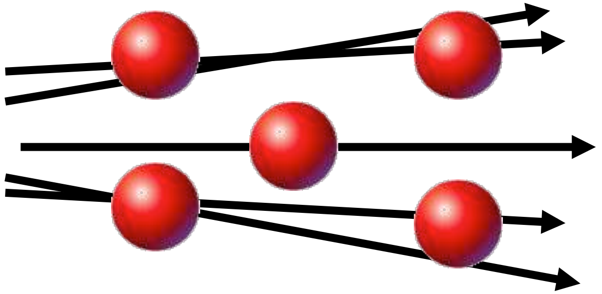
$$\begin{cases} x \\ x' = \dfrac{dx}{dz} = \dfrac{p_x}{p_z} \end{cases} \qquad p_x << p_z$$

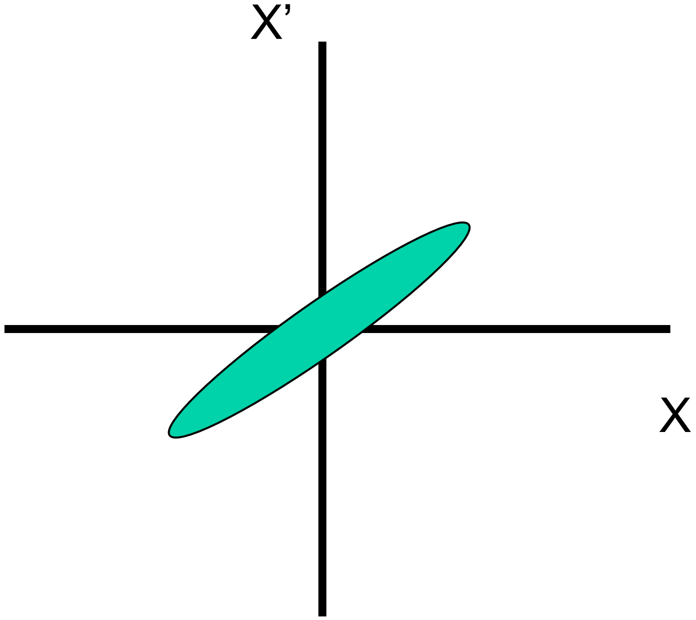


Trace space of a laminar beam



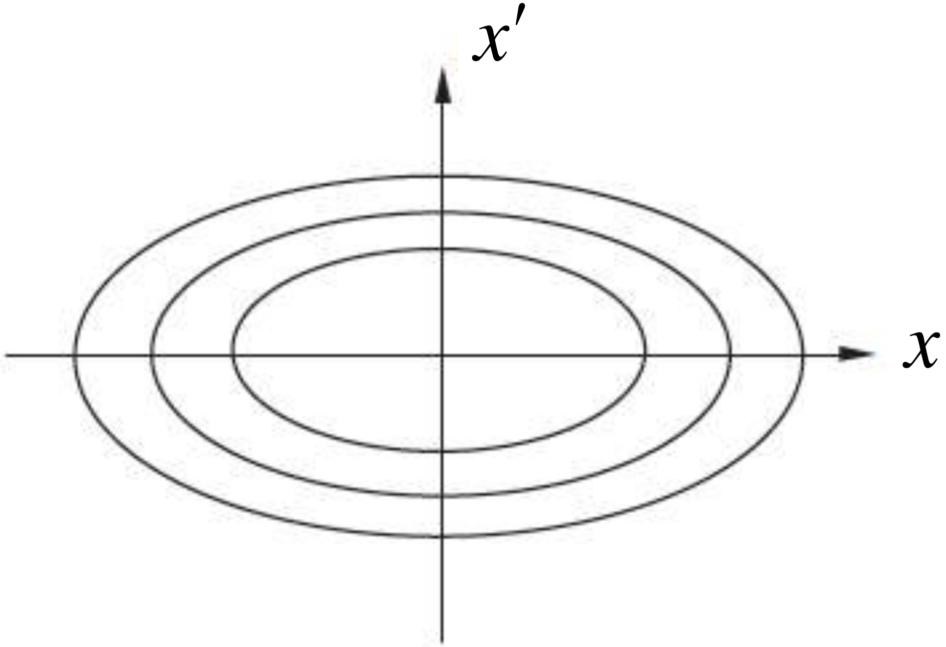
Trace space of non laminar beam





In a system where all the forces acting on the particles are linear (i.e., proportional to the particle's displacement x from the beam axis), it is useful to assume an elliptical shape for the area occupied by the beam in x-x' trace space.

$$\boxed{\ddot{x} + kx = 0}$$



$$H = \frac{1}{2m}[p_x^2 + m^2\omega^2 x^2]$$

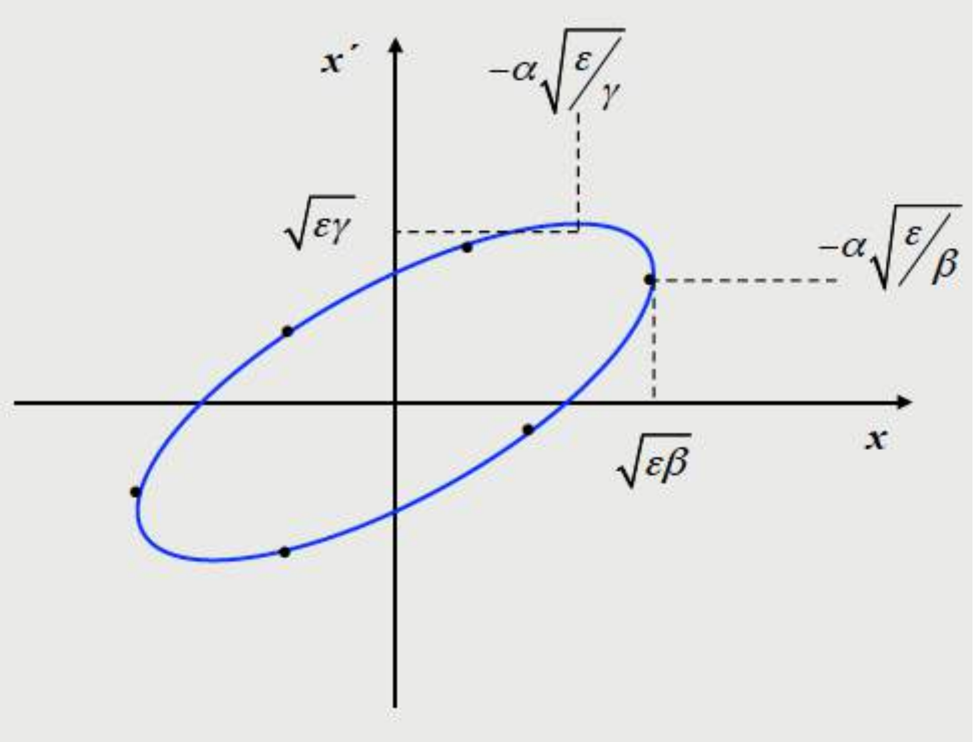
$$\dot{x}_i = \frac{\partial H}{\partial p_i}, \qquad \dot{p}_i = -\frac{\partial H}{\partial x_i},$$

Geometric emittance: $\boxed{\varepsilon_g}$

Ellipse equation: $\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon_g$

Twiss parameters: $\beta\gamma - \alpha^2 = 1$ $\beta' = -2\alpha$

Ellipse area: $A = \pi\varepsilon_g$



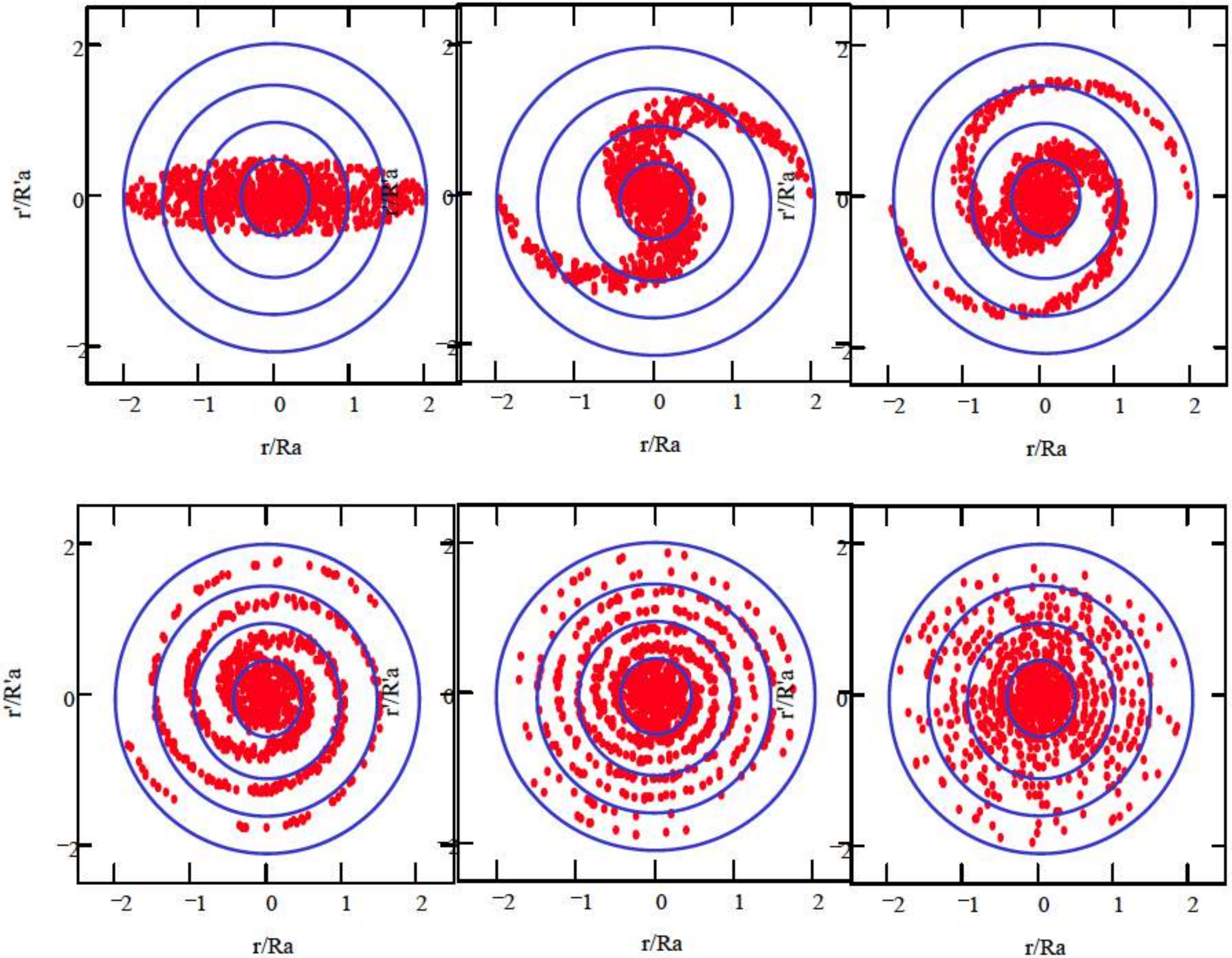
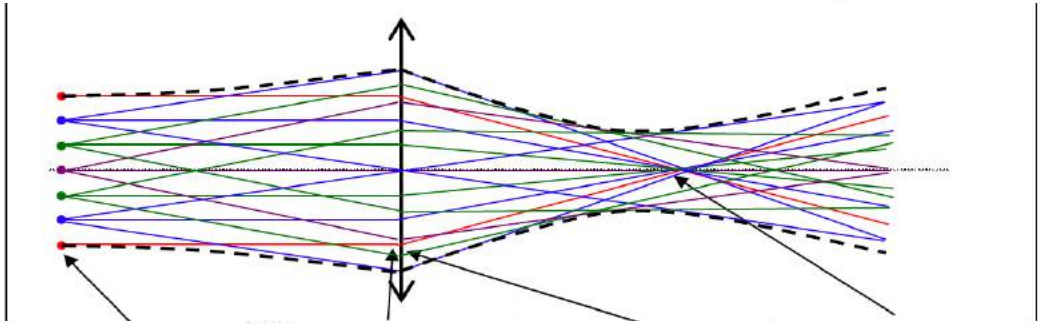
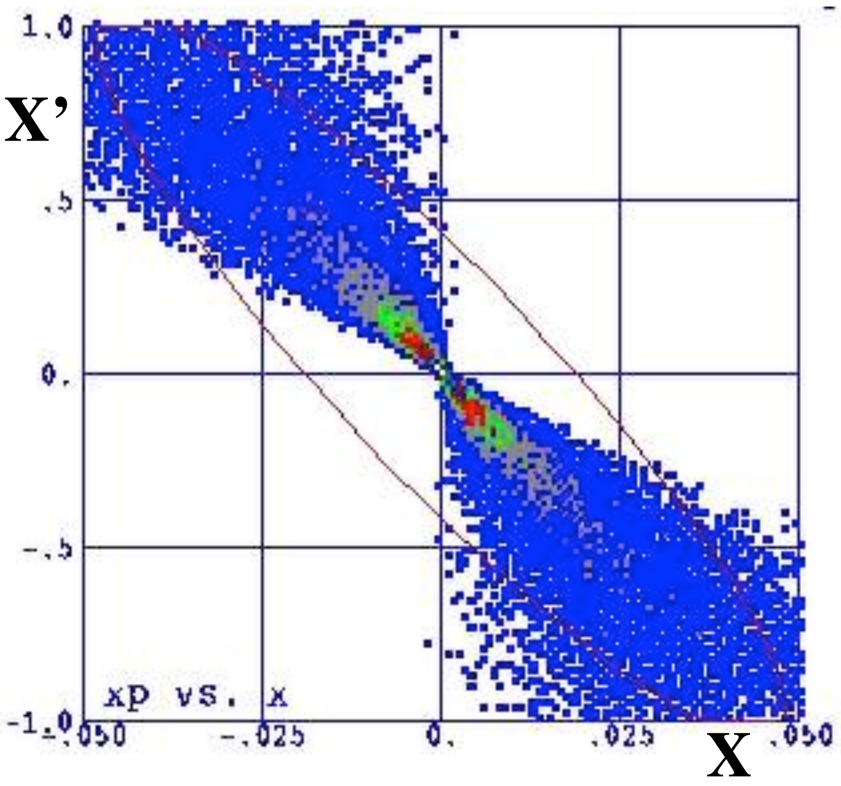


Fig. 17: Filamentation of mismatched beam in non-linear force

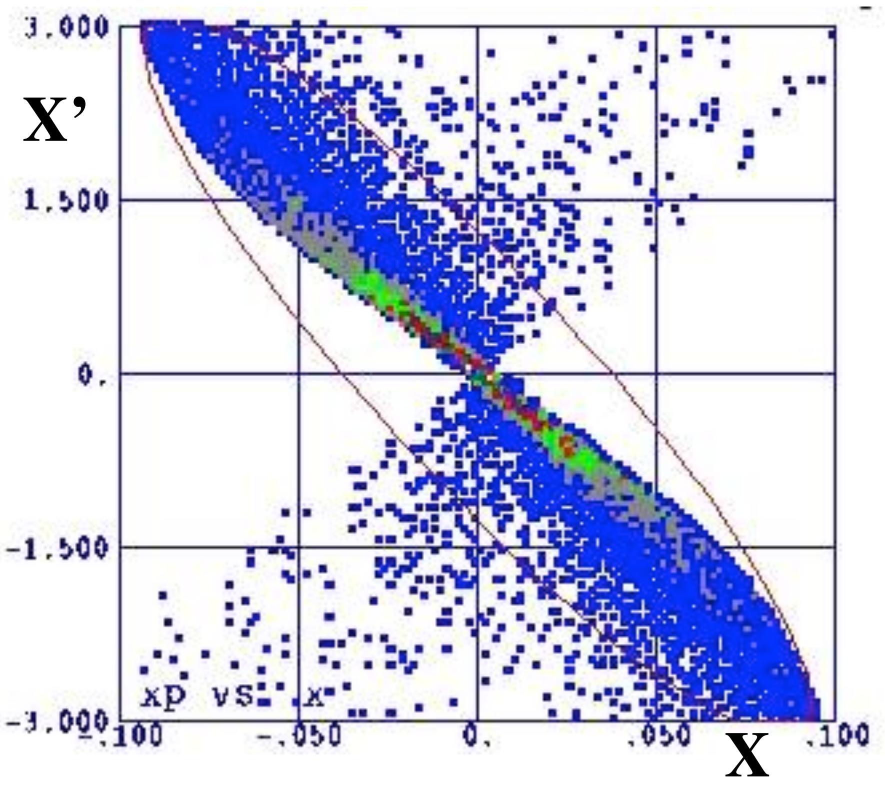
Trace space evolution



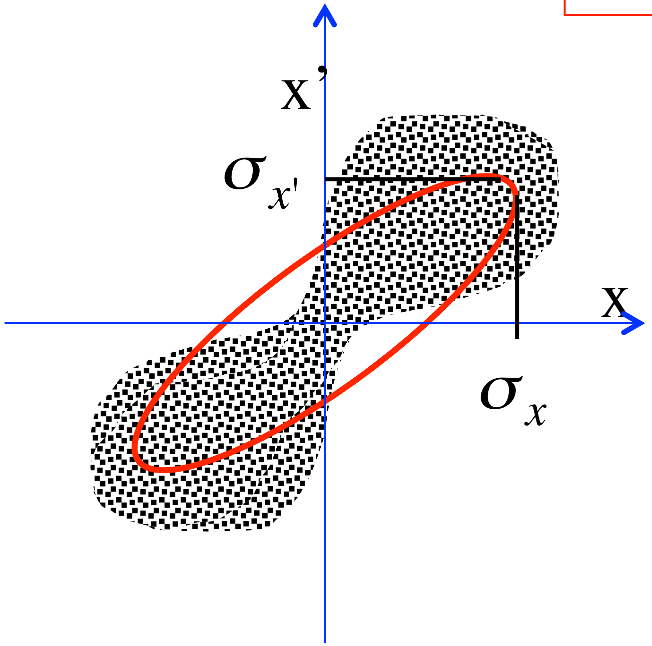
No space charge => cross over



With space charge => no cross over



rms emittance $\boxed{\varepsilon_{rms}}$



$$\int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,x')\,dx\,dx' = 1 \qquad \boxed{f'(x,x')=0}$$

rms beam envelope:

$$\sigma_x^2 = \langle x^2 \rangle = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} x^2 f(x,x')\,dx\,dx'$$

Define rms emittance:

$$\boxed{\gamma x^2 + 2\alpha x x' + \beta x'^2 = \varepsilon_{rms}}$$

such that:

$$\sigma_x = \sqrt{\langle x^2 \rangle} = \sqrt{\beta \varepsilon_{rms}}$$

$$\sigma_{x'} = \sqrt{\langle x'^2 \rangle} = \sqrt{\gamma \varepsilon_{rms}}$$

$$\boxed{\begin{aligned}\sigma_x &= \sqrt{\langle x^2 \rangle} = \sqrt{\beta \varepsilon_{rms}} \\ \sigma_{x'} &= \sqrt{\langle x'^2 \rangle} = \sqrt{\gamma \varepsilon_{rms}} \\ \sigma_{xx'} &= \langle xx' \rangle = -\alpha \varepsilon_{rms}\end{aligned}}$$

$$\alpha = -\frac{\beta'}{2}$$

It holds also the relation: $\gamma\beta - \alpha^2 = 1$

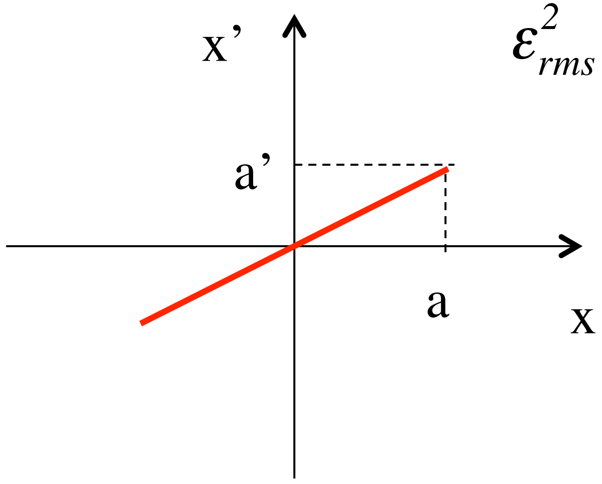
Substituting α, β, γ we get $\frac{\sigma_{x'}^2}{\varepsilon_{rms}} \frac{\sigma_x^2}{\varepsilon_{rms}} - \left(\frac{\sigma_{xx'}}{\varepsilon_{rms}}\right)^2 = 1$

We end up with the definition of rms emittance in terms of the second moments of the distribution:

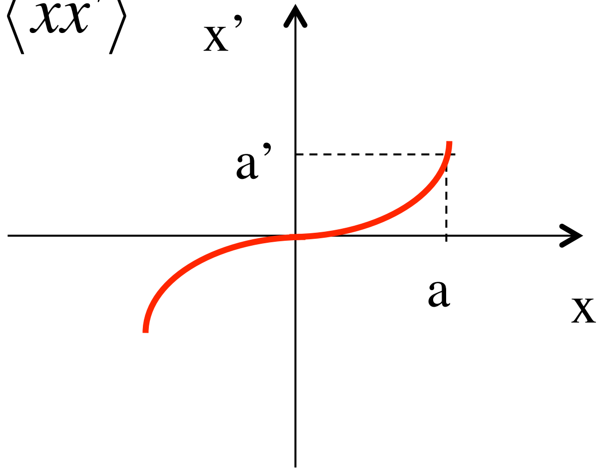
$$\boxed{\varepsilon_{rms} = \sqrt{\sigma_x^2 \sigma_{x'}^2 - \sigma_{xx'}^2} = \sqrt{\left(\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2\right)}}$$

$$x' = \frac{p_x}{p_z}$$

What does rms emittance tell us about phase space distributions under linear or non-linear forces acting on the beam?



$$\varepsilon_{rms}^2 = \langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2$$



Assuming a generic x, x' correlation of the type: $x' = Cx^n$

$$\varepsilon_{rms}^2 = C^2 \left(\langle x^2 \rangle \langle x^{2n} \rangle - \langle x^{n+1} \rangle^2 \right)$$

When n = 1 ==> $\varepsilon_{rms} = 0$

When n $\neq$ 1 ==> $\varepsilon_{rms} \neq 0$

Normalized rms emittance: $\varepsilon_{n,rms}$

Canonical transverse momentum: $p_x = p_z x' = m_o c\beta\gamma x'$ $p_z \approx p$

$$\boxed{\varepsilon_{n,rms} = \sqrt{\sigma_x^2\sigma_{p_x}^2 - \sigma_{xp_x}^2} = \frac{1}{m_o c}\sqrt{\left(\left\langle x^2\right\rangle\left\langle p_x^2\right\rangle - \left\langle xp_x\right\rangle^2\right)}}$$

Liouville theorem: the density of particles *n*, or the volume V occupied by a given number of particles in phase space (x,p_x,y,p_y,z,p_z) remains invariant.

$$\boxed{\frac{dn}{dt} = 0}$$

It hold also in the projected phase spaces $(x,p_x),(y,p_y)(,z,p_z)$ provided that there are no couplings

Limit of single particle emittance

Limits are set by Quantum Mechanics on the knowledge of the two conjugate variables (x,p_x). According to Heisenberg:

$$\boxed{\sigma_x \sigma_{p_x} \geq \frac{\hbar}{2}}$$

This limitation can be expressed by saying that the state of a particle is not exactly represented by a point, but by a small uncertainty volume of the order of $\hbar^3$ in the 6D phase space.

In particular for a single electron in 2D phase space it holds:

$$\boxed{\varepsilon_{n,rms} = \frac{1}{m_o c}\sqrt{\sigma_x^2 \sigma_{p_x}^2 - \sigma_{xp_x}^2} \Rightarrow \begin{cases} = 0 & \text{classical limit} \\ \geq \dfrac{1}{2}\dfrac{\hbar}{m_o c} = \dfrac{\lambda_c}{2} = 1.9 \times 10^{-13} m & \text{quantum limit} \end{cases}}$$

Where $\boxed{\lambda_c}$ is the reduced Compton wavelength.

Normalized and un-normalized emittances

$$p_x = p_z x' = m_o c \beta\gamma x'$$

$$\varepsilon_{n,rms} = \frac{1}{m_o c}\sqrt{\left(\left\langle x^2\right\rangle\left\langle p_x^2\right\rangle - \left\langle x p_x\right\rangle^2\right)} = \sqrt{\left(\left\langle x^2\right\rangle\left\langle\left(\beta\gamma x'\right)^2\right\rangle - \left\langle x\beta\gamma x'\right\rangle^2\right)} = \left\langle\beta\gamma\right\rangle\varepsilon_{rms}$$

Assuming **small energy** spread within the beam, the normalized and un-normalized emittances can be related by the above approximated relation.

This approximation that is often used in conventional accelerators may be strongly misleading when adopted to describe beams with significant energy spread, as the one at present produced by plasma accelerators.

When the correlations between the energy and transverse positions are negligible (as in a drift without collective effects) we can write:

$$\varepsilon_{n,rms}^2 = \left\langle \beta^2\gamma^2 \right\rangle \left\langle x^2 \right\rangle \left\langle x'^2 \right\rangle - \left\langle \beta\gamma \right\rangle^2 \left\langle xx' \right\rangle^2$$

Considering now the definition of relative energy spread:

$$\sigma_\gamma^2 = \frac{\left\langle \beta^2\gamma^2 \right\rangle - \left\langle \beta\gamma \right\rangle^2}{\left\langle \beta\gamma \right\rangle^2}$$

which can be inserted in the emittance definition to give:

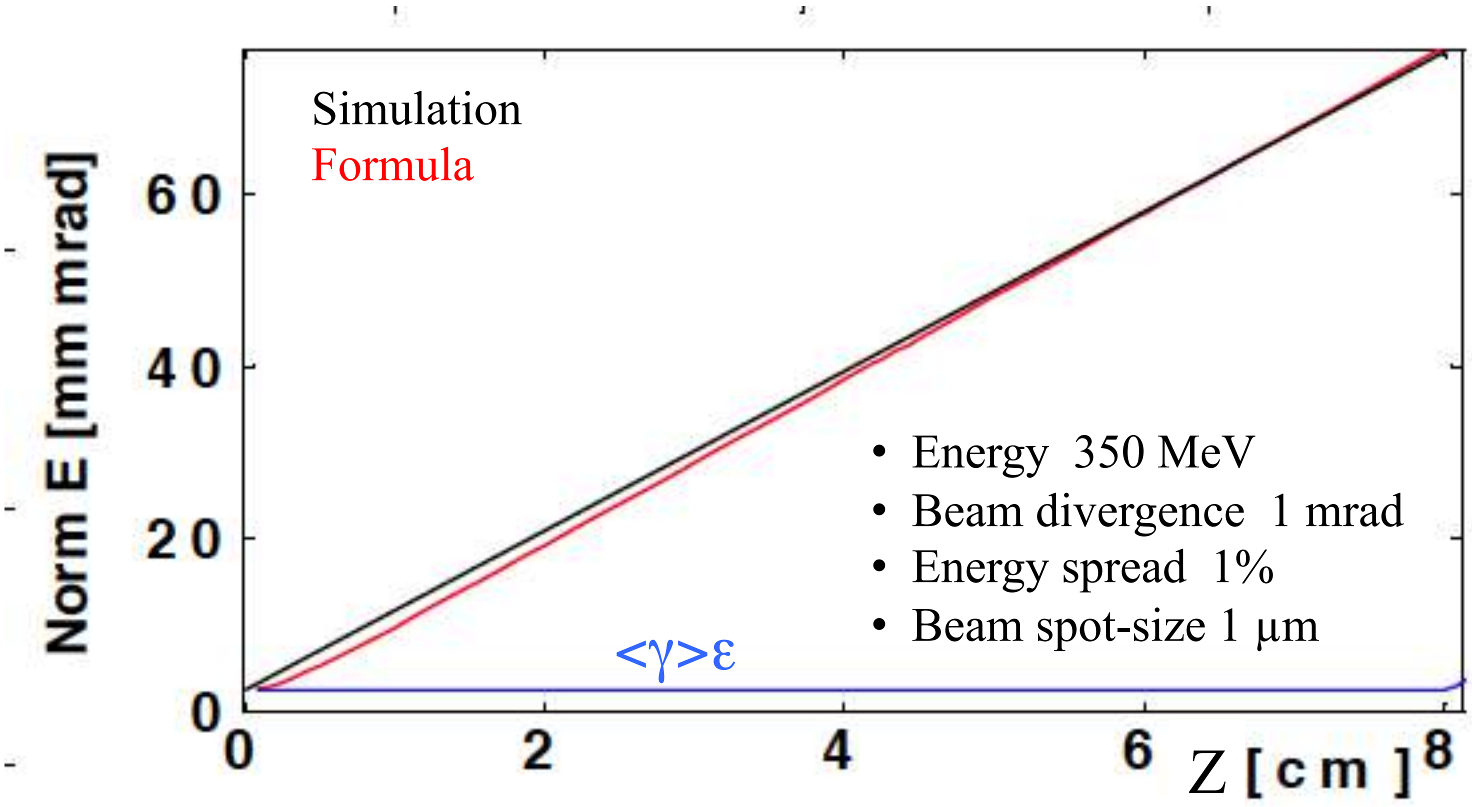
$$\varepsilon_{n,rms}^2 = \left\langle \beta^2\gamma^2 \right\rangle \sigma_\gamma^2 \left\langle x^2 \right\rangle \left\langle x'^2 \right\rangle + \left\langle \beta\gamma \right\rangle^2 \left(\left\langle x^2 \right\rangle \left\langle x'^2 \right\rangle - \left\langle xx' \right\rangle^2 \right)$$

Assuming relativistic electrons (β=1) we get:

$$\boxed{\varepsilon_{n,rms}^2 = \left\langle \gamma^2 \right\rangle \left(\sigma_\gamma^2 \sigma_x^2 \sigma_{x'}^2 + \varepsilon_{rms}^2 \right)}$$

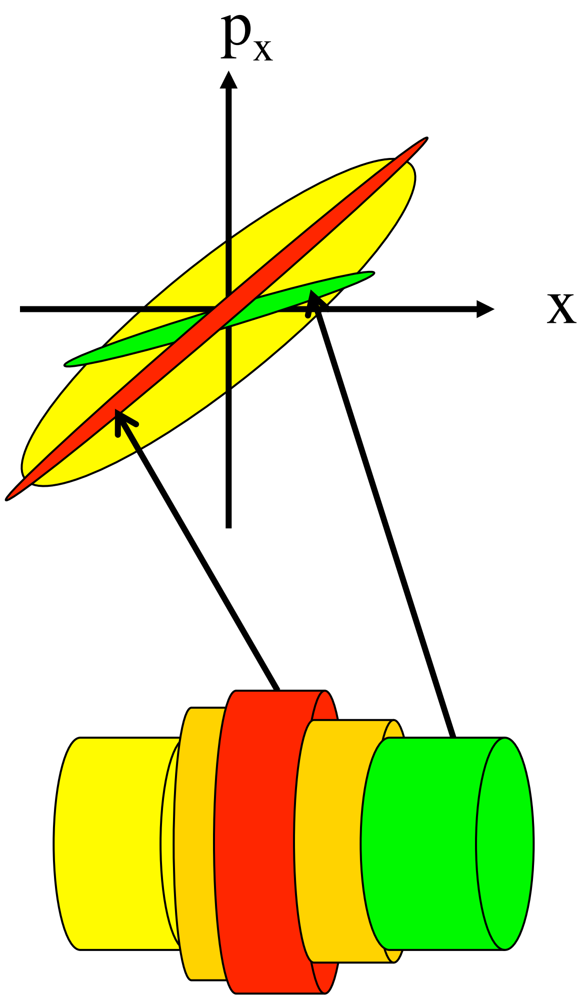
$$\boxed{\varepsilon_{n,rms}^{2} = \langle\gamma^{2}\rangle\left(\sigma_{\gamma}^{2}\sigma_{x}^{2}\sigma_{x'}^{2} + \varepsilon_{rms}^{2}\right) = \langle\gamma^{2}\rangle\left(\sigma_{\gamma}^{2}\sigma_{o,x'}^{4}\left(z - z_{o}\right)^{2} + \varepsilon_{rms}^{2}\right)}$$

showing that beams with large energy spread an divergence undergo a significant normalized emittance growth even in a drift



- Energy 350 MeV
- Beam divergence 1 mrad
- Energy spread 1%
- Beam spot-size 1 µm

Phase space, slice emittance and longitudinal correlations



Projected Phase Space

Slice Phase Spaces

FEL cooperation length

$$L_c = \lambda_r / 4\pi\rho,$$

$$\boxed{\begin{aligned}\sigma_x &= \sqrt{\langle x^2 \rangle} = \sqrt{\beta\varepsilon_{rms}} \\ \sigma'_x &= \sqrt{\langle x'^2 \rangle} = \sqrt{\gamma\varepsilon_{rms}} \\ \sigma_{xx'} &= \langle xx' \rangle = -\alpha\varepsilon_{rms}\end{aligned}}$$

It holds also the relation: $\gamma\beta - \alpha^2 = 1$

Substituting α, β, γ we get $\frac{\sigma_{x'}^2}{\varepsilon_{rms}} \frac{\sigma_x^2}{\varepsilon_{rms}} - \left(\frac{\sigma_{xx'}}{\varepsilon_{rms}}\right)^2 = 1$

We end up with the definition of rms emittance in terms of the second moments of the distribution:

$$\boxed{\varepsilon_{rms} = \sqrt{\sigma_x^2\sigma_{x'}^2 - \sigma_{xx'}^2} = \sqrt{\left(\langle x^2 \rangle \langle x'^2 \rangle - \langle xx' \rangle^2\right)}}$$

$$x' = \frac{p_x}{p_z}$$

Envelope Equation without Acceleration

Now take the derivatives:

$$\boxed{\frac{d\sigma_x}{dz}} = \frac{d}{dz}\sqrt{\left\langle x^2 \right\rangle} = \frac{1}{2\sigma_x}\frac{d}{dz}\left\langle x^2 \right\rangle = \frac{1}{2\sigma_x}2\left\langle xx' \right\rangle = \boxed{\frac{\sigma_{xx'}}{\sigma_x}}$$

$$\boxed{\frac{d^2\sigma_x}{dz^2}} = \frac{d}{dz}\frac{\sigma_{xx'}}{\sigma_x} = \frac{1}{\sigma_x}\frac{d\sigma_{xx'}}{dz} - \frac{\sigma_{xx'}^2}{\sigma_x^3} = \frac{1}{\sigma_x}\left(\left\langle x'^2 \right\rangle + \left\langle xx' \right\rangle\right) - \frac{\sigma_{xx'}^2}{\sigma_x^3} = \boxed{\frac{\sigma_{x'}^2 + \left\langle xx'' \right\rangle}{\sigma_x} - \frac{\sigma_{xx'}^2}{\sigma_x^3}}$$

And simplify:

$$\boxed{\sigma_x'' =} \frac{\sigma_x^2\sigma_{x'}^2 - \sigma_{xx'}^2}{\sigma_x^3} + \frac{\left\langle xx'' \right\rangle}{\sigma_x} = \boxed{\frac{\varepsilon_{rms}^2}{\sigma_x^3} + \frac{\left\langle xx'' \right\rangle}{\sigma_x}}$$

We obtain the rms envelope equation in which the rms emittance enters as defocusing pressure like term.

$$\boxed{\sigma_x'' - \frac{\left\langle xx'' \right\rangle}{\sigma_x} = \frac{\varepsilon_{rms}^2}{\sigma_x^3}}$$

$$\boxed{\frac{\varepsilon_{rms}^2}{\sigma_x^3} \approx \frac{T}{V} \approx P}$$

Beam Thermodynamics

Kinetic theory of gases defines temperatures in each directions and global as:

$$k_B T_x = m\left\langle v_x^2 \right\rangle \qquad T = \frac{1}{3}\left(T_x + T_y + T_z\right) \qquad E_k = \frac{1}{2}m\left\langle v^2 \right\rangle = \frac{3}{2}k_B T$$

Definition of beam temperature in analogy:

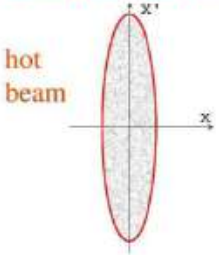
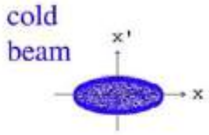
$$k_B T_{beam,x} = \gamma m_o \left\langle v_x^2 \right\rangle \qquad \left\langle v_x^2 \right\rangle = \beta^2 c^2 \left\langle x'^2 \right\rangle = \beta^2 c^2 \sigma_{x'}^2 = \beta^2 c^2 \frac{\varepsilon_{rms}^2}{\sigma_x^2}$$

$$\boxed{k_B T_{beam,x} = \gamma m_o \left\langle v_x^2 \right\rangle = \gamma m_o \beta^2 c^2 \frac{\varepsilon_{rms}^2}{\sigma_x^2}}$$

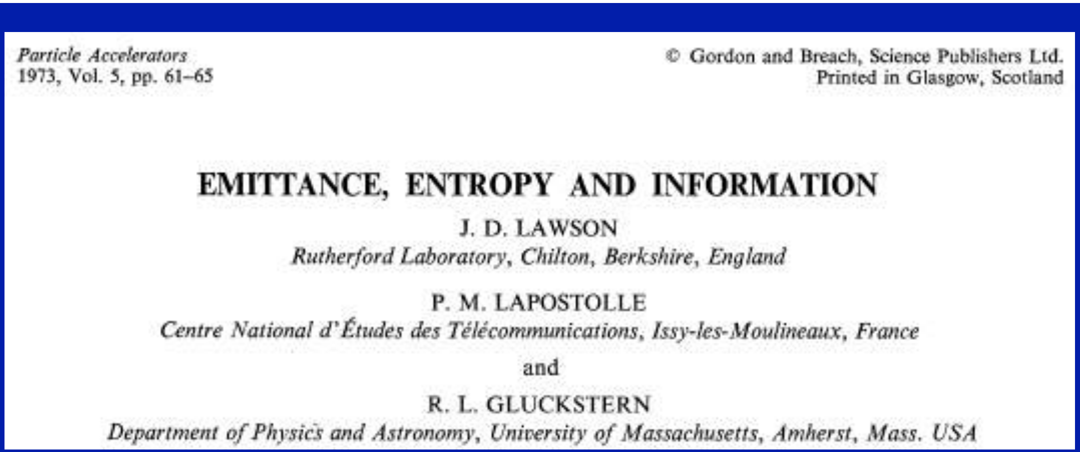
We get:

$$\boxed{\begin{aligned} &P_{beam,x} = n k_B T_{beam,x} = n\gamma m_o \beta^2 c^2 \frac{\varepsilon_{rms}^2}{\sigma_x^2} = N_T \gamma m_o \beta^2 c^2 \frac{\varepsilon_{rms}^2}{\sigma_L \sigma_x^2} \\ &n = \frac{N}{\pi \sigma_L \sigma_x^2} = \frac{N_T}{\sigma_L} \end{aligned}}$$

$$k_B T_{beam,x} = \gamma m_o \beta^2 c^2 \frac{\varepsilon_{rms}}{\beta_x}$$

Property	Hot beam	Cold beam
ion mass (m_o)	**heavy ion**	**light ion**
ion energy ($\beta\gamma$)	**high energy**	**low energy**
beam emittance (ε)	**large emittance**	**small emittance**
lattice properties ($\gamma_{x,y} \approx 1/\beta_{x,y}$)	**strong focus (low β)**	**high β**
phase space portrait		

Electron Cooling: Temperature relaxation by mixing a hot ion beam with co-moving cold (light) electron beam.



Particle Accelerators
1973, Vol. 5, pp. 61–65

EMITTANCE, ENTROPY AND INFORMATION

J. D. LAWSON
Rutherford Laboratory, Chilton, Berkshire, England

P. M. LAPOSTOLLE
Centre National d'Études des Télécommunications, Issy-les-Moulineaux, France

and

R. L. GLUCKSTERN
Department of Physics and Astronomy, University of Massachusetts, Amherst, Mass. USA

$$S = kN\log\left(\pi\varepsilon\right)$$

Envelope Equation with Linear Focusing

$$\boxed{\sigma_x'' - \frac{\langle xx'' \rangle}{\sigma_x} = \frac{\varepsilon_{rms}^2}{\sigma_x^3}}$$

Assuming that each particle is subject only to a linear focusing force, without acceleration: $x'' + k_x^2 x = 0$

take the average over the entire particle ensemble $\langle xx'' \rangle = -k_x^2 \langle x^2 \rangle$

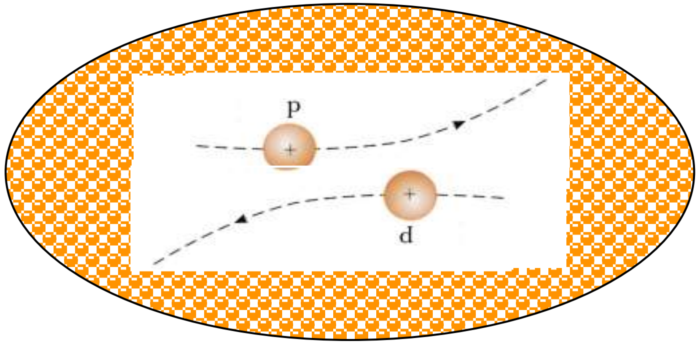
$$\boxed{\sigma_x'' + k_x^2 \sigma_x = \frac{\varepsilon_{rms}^2}{\sigma_x^3}}$$

We obtain the rms envelope equation with a linear focusing force in which, unlike in the single particle equation of motion, the rms emittance enters as defocusing pressure like term.

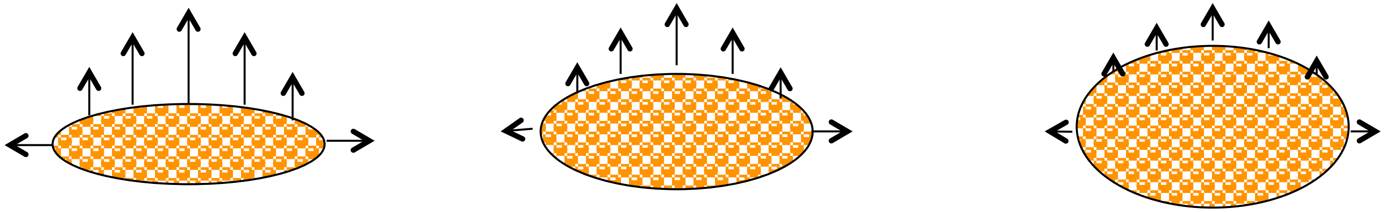
Space Charge: What does it mean?

The net effect of the **Coulomb** interactions in a multi-particle system can be classified into two regimes:

1) **Collisional Regime** ==> dominated by **binary collisions** caused by close particle encounters ==> **Single Particle Effects**



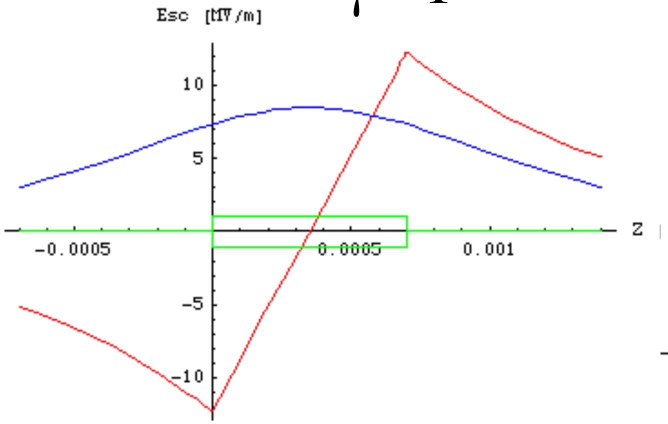
2) **Space Charge Regime** ==> dominated by the **self field** produced by the particle distribution, which varies appreciably only over large distances compare to the average separation of the particles ==> **Collective Effects**



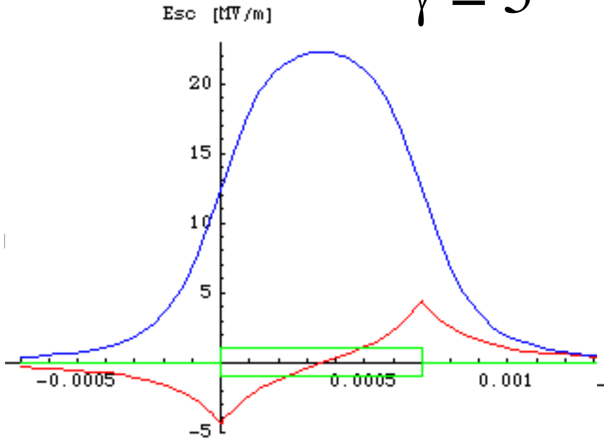
$$\boxed{E_z(0,s,\gamma) = \frac{I}{2\pi\gamma\varepsilon_0 R^2 \beta c} h(s,\gamma)}$$

$$\boxed{E_r(r,s,\gamma) = \frac{Ir}{2\pi\varepsilon_0 R^2 \beta c} g(s,\gamma)}$$

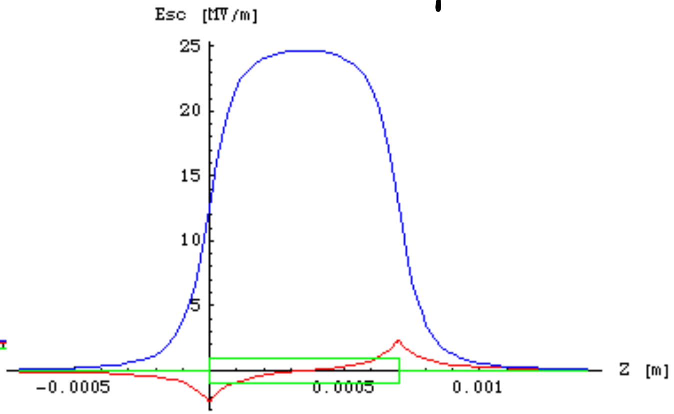
$\gamma = 1$



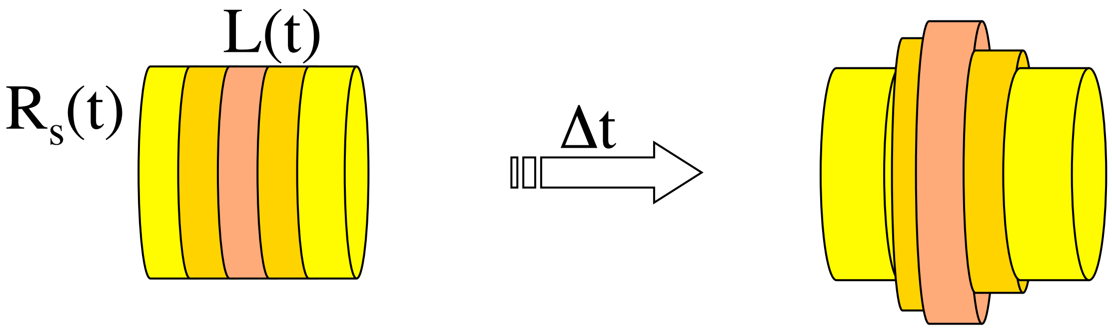
$\gamma = 5$



$\gamma = 10$



$$\boxed{F_r = \frac{eE_r}{\gamma^2} = \frac{eIr}{2\pi\gamma^2\varepsilon_0 R^2 \beta c} g(s,\gamma)}$$



$$\boxed{B_\vartheta = \frac{\beta}{c} E_r}$$

Lorentz Force

$$\boxed{E_r(r,s,\gamma) = \frac{Ir}{2\pi\varepsilon_0 R^2 \beta c} g(s,\gamma)}$$

$$\boxed{F_r = e\left(E_r - \beta c B_\vartheta\right) = e\left(1 - \beta^2\right) E_r = \frac{eE_r}{\gamma^2}}$$

is a **linear** function of the transverse coordinate

$$\boxed{\frac{dp_r}{dt} = F_r = \frac{eE_r}{\gamma^2} = \frac{eIr}{2\pi\gamma^2\varepsilon_0 R^2 \beta c} g(s,\gamma)}$$

The attractive magnetic force , which becomes significant at high velocities, tends to compensate for the repulsive electric force. Therefore space charge defocusing is primarily a non-relativistic effect.

$$\boxed{F_x = \frac{eIx}{2\pi\gamma^2\varepsilon_0\sigma_x^2\beta c} g(s,\gamma)}$$

$$k_{sc}(s,\gamma) = \frac{2I}{I_A(\beta\gamma)^3} g(s,\gamma)$$

Envelope Equation with Space Charge

Single particle transverse motion:

$$\frac{dp_x}{dt} = F_x \qquad p_x = p\ x' = \beta\gamma m_o c x'$$

$$\frac{d}{dt}\left(px'\right) = \beta c \frac{d}{dz}\left(p\ x'\right) = F_x$$

$$x'' = \frac{F_x}{\beta c p}$$

$$\boxed{x'' = \frac{k_{sc}\left(s,\gamma\right)}{\sigma_x^2} x}$$

Space Charge de-focusing force

Generalized perveance

$$k_{sc}\left(s,\gamma\right) = \frac{2I}{I_A\left(\beta\gamma\right)^3} g\left(s,\gamma\right)$$

$$I_A = \frac{4\pi\varepsilon_o m_o c^3}{e} = 17kA$$

Now we can calculate the term $\langle xx''\rangle$ that enters in the envelope equation

$$\boxed{\sigma_x'' = \frac{\varepsilon_{rms}^2}{\sigma_x^3} - \frac{\langle xx''\rangle}{\sigma_x}} \qquad \langle xx''\rangle = \frac{k_{sc}}{\sigma_x^2}\langle x^2\rangle = k_{sc}$$

Including all the other terms the envelope equation reads:

Space Charge De-focusing Force

$$\boxed{\sigma_x'' + k^2\sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2\sigma_x^3} + \frac{k_{sc}}{\sigma_x}}$$

Emittance Pressure

External Focusing Forces

Laminarity Parameter: $\boxed{\rho = \frac{(\beta\gamma)^2 k_{sc}\sigma_x^2}{\varepsilon_n^2}}$

The beam undergoes two regimes along the accelerator

$$\sigma''_x + k^2\sigma_x = \cancel{\frac{\varepsilon_n^2}{(\beta\gamma)^2\sigma_x^3}} + \frac{k_{sc}}{\sigma_x}$$

ρ>>1

Laminar Beam

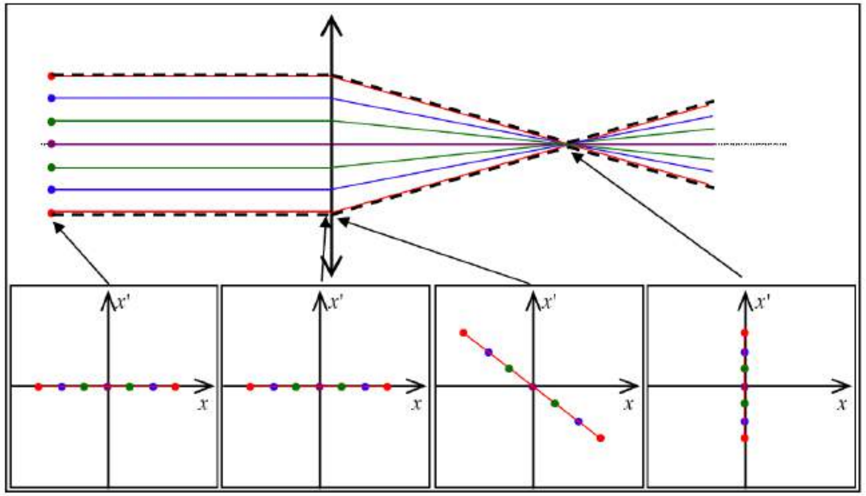


Fig. 10: Particle trajectories in laminar beam

$$\sigma''_x + k^2\sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2\sigma_x^3} + \cancel{\frac{k_{sc}}{\sigma_x}}$$

ρ<<1

Thermal Beam

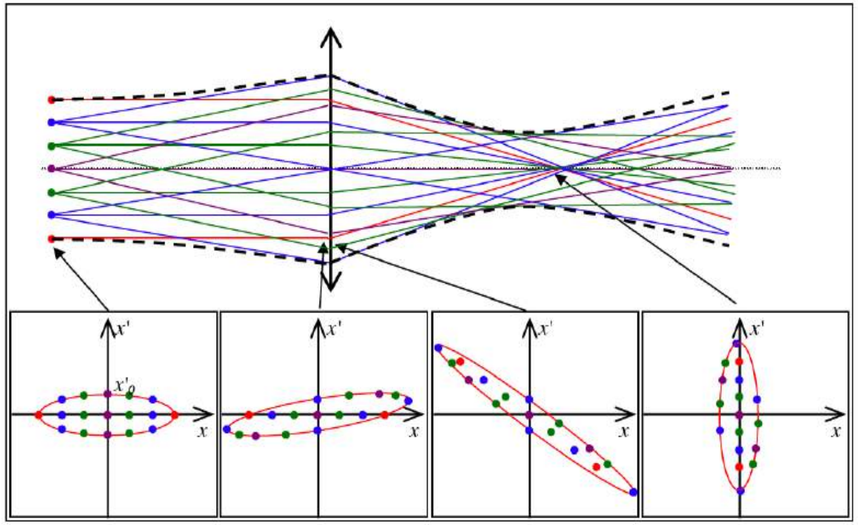


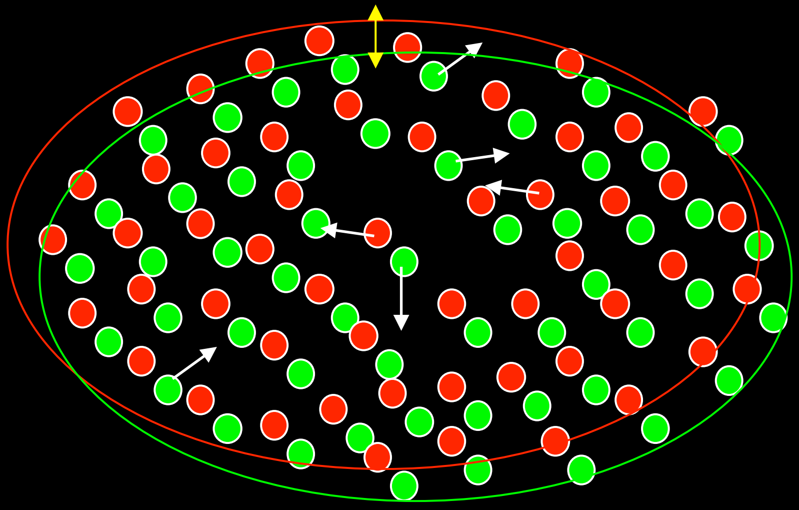
Fig. 11: Particle trajectories in non-zero emittance beam

Space Charge induced emittance oscillations in a laminar beam

Surface charge density

$$\sigma = e\,n\,\delta x$$

$$\delta x$$



Surface electric field

$$E_x = -\sigma/\epsilon_0 = -e\,n\,\delta x/\epsilon_0$$

Restoring force

$$m\,\frac{d^2\delta x}{dt^2} = e\,E_x = -m\,\omega_p{}^2\,\delta x$$

Plasma frequency

$$\omega_p{}^2 = \frac{n\,e^2}{\epsilon_0\,m}$$

Plasma oscillations

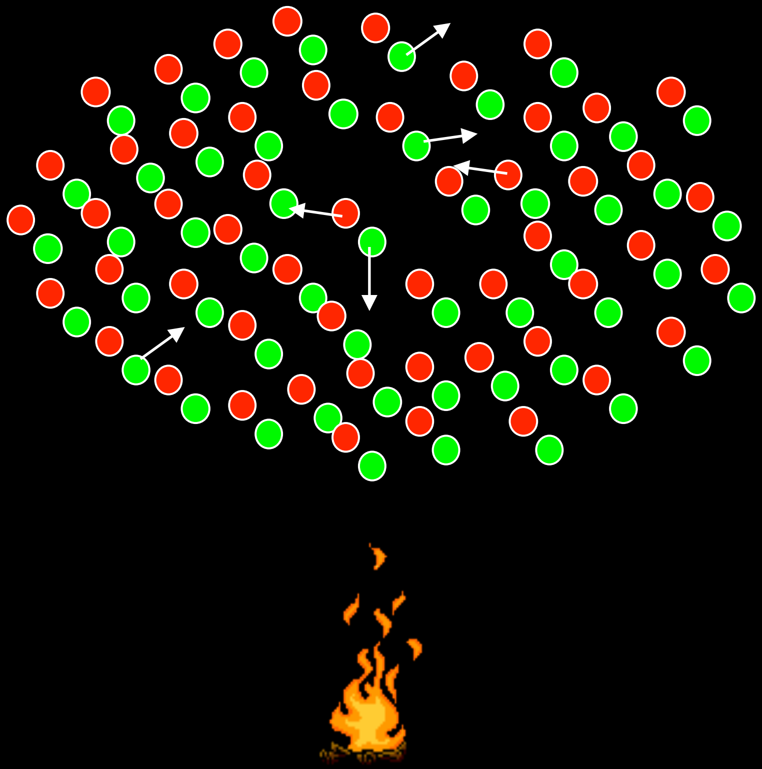
$$\delta x = (\delta x)_0\,\cos\left(\omega_p\,t\right)$$

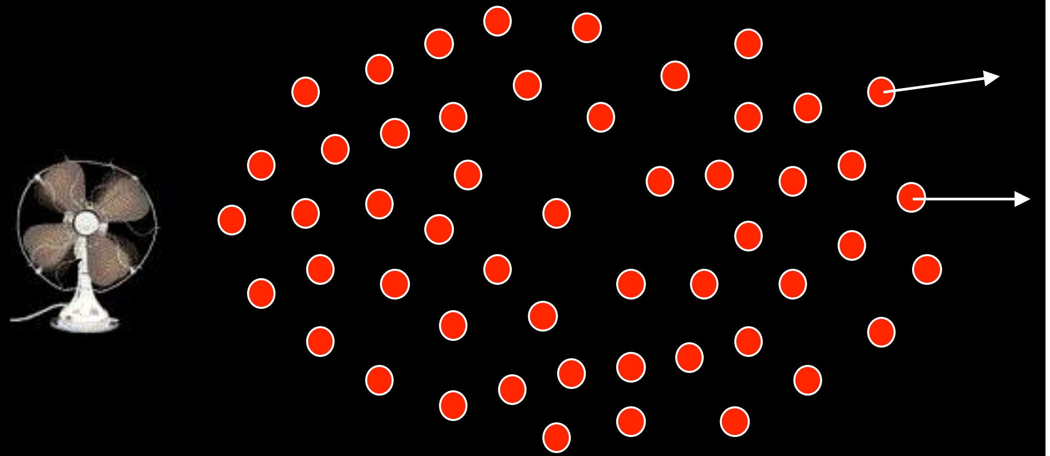
Neutral Plasma

- Oscillations
- Instabilities
- EM Wave propagation

Single Component Cold Relativistic Plasma

Magnetic focusing





Magnetic focusing

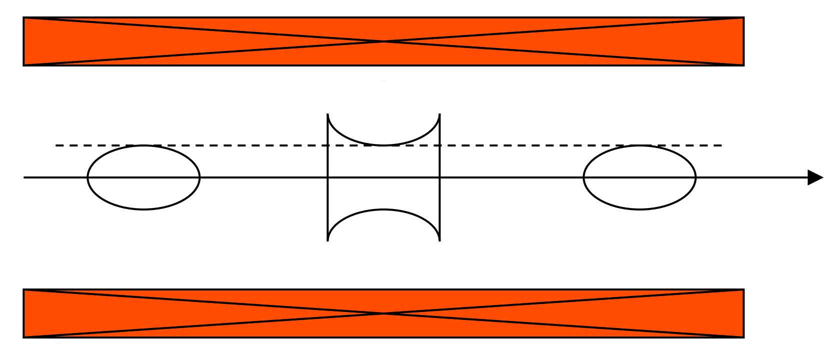
$$\sigma'' + k_s^2\sigma = \frac{k_{sc}(s,\gamma)}{\sigma}$$

Single Component Relativistic Plasma

$$k_s = \frac{qB}{2mc\beta\gamma}$$

Equilibrium solution:

$$\sigma_{eq}(s,\gamma) = \frac{\sqrt{k_{sc}(s,\gamma)}}{k_s}$$



Small perturbation:

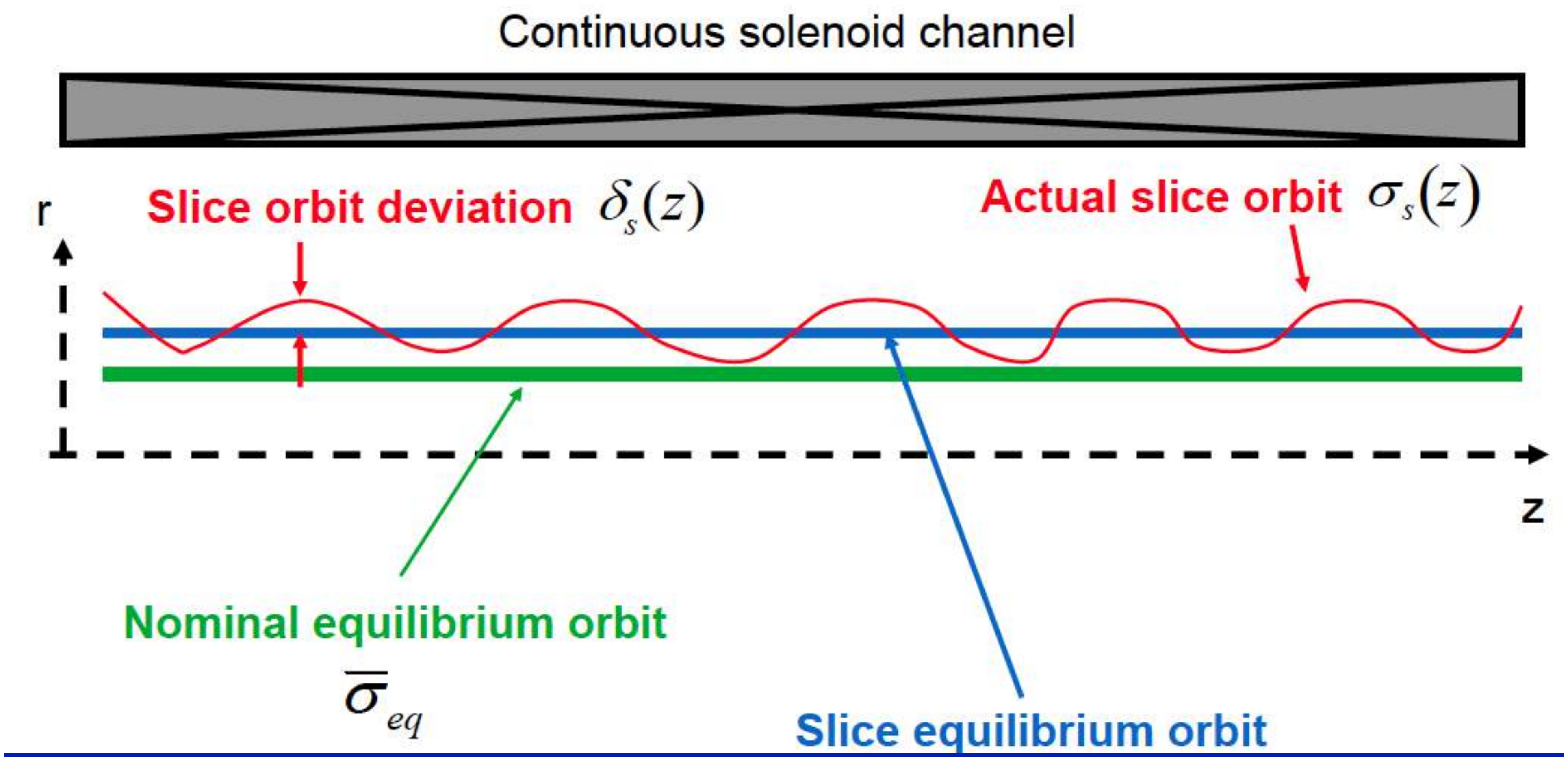
$$\sigma(\zeta) = \sigma_{eq}(s) + \delta\sigma(s)$$

$$\delta\sigma''(s) + 2k_s^2\delta\sigma(s) = 0$$

$$\delta\sigma(s) = \delta\sigma_o(s)\,cos\left(\sqrt{2}k_s z\right)$$

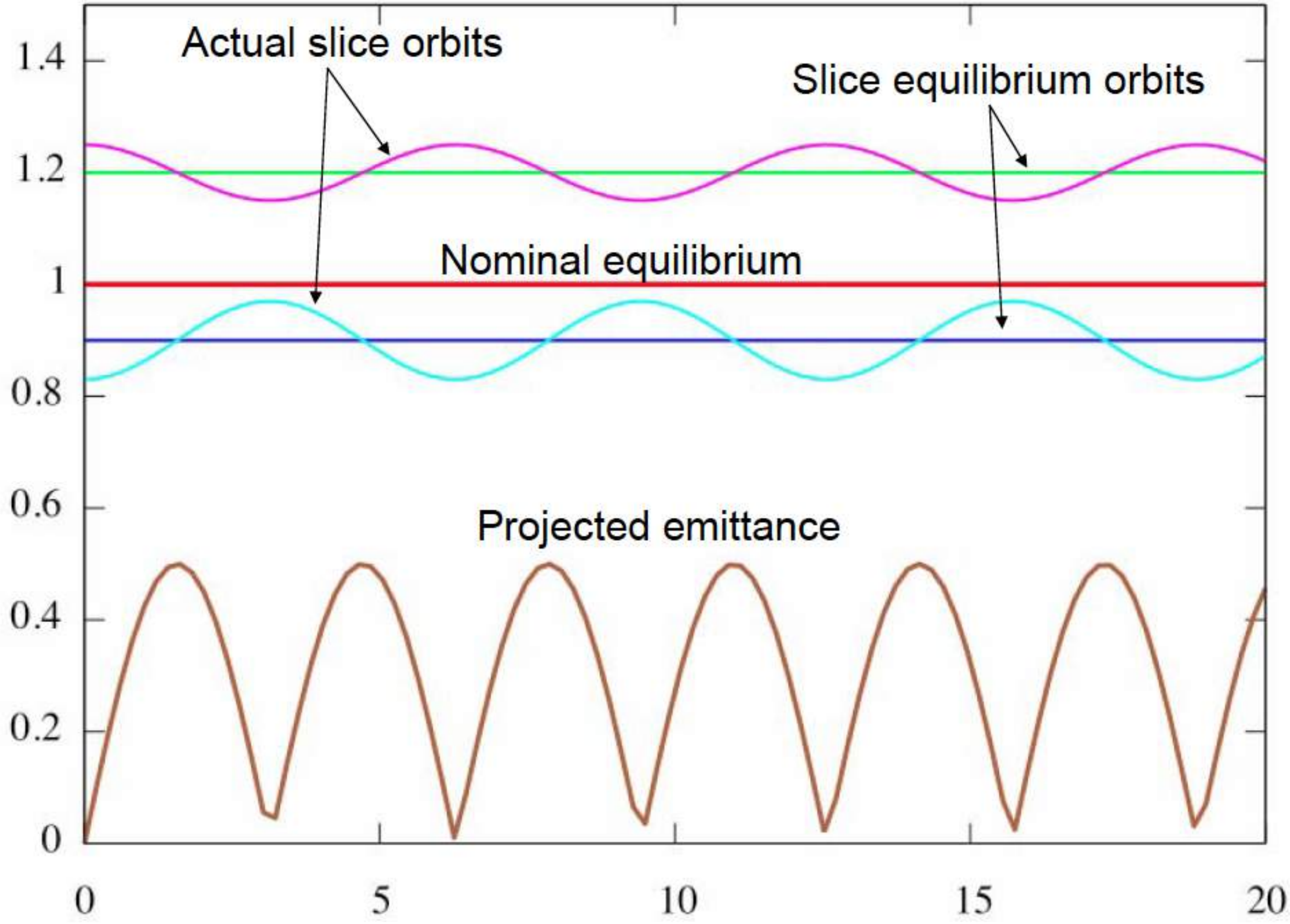
Perturbed trajectories oscillate around the equilibrium with the same frequency but with different amplitudes:

$$\sigma(s) = \sigma_{eq}(s) + \delta\sigma_o(s)\,cos\left(\sqrt{2}k_s z\right)$$



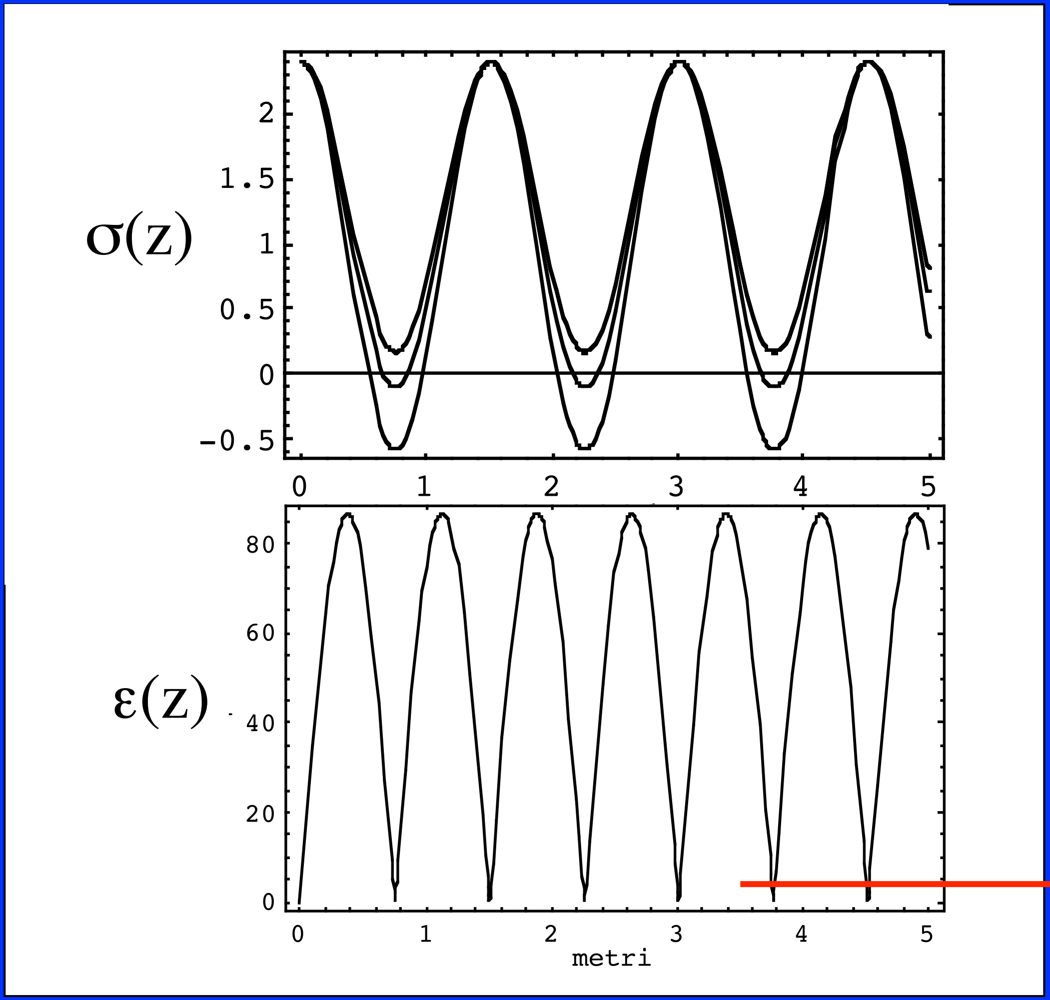
Perturbed trajectories oscillate around the equilibrium with the same frequency but with different amplitudes:

$$\sigma(s) = \sigma_{eq}(s) + \delta\sigma_o(s)\, cos\left(\sqrt{2}k_s z\right)$$



$$\varepsilon_{rms} = \sqrt{\sigma_x^2\sigma_{x'}^2 - \sigma_{xx'}^2} \approx \left|\sin\left(\sqrt{2}k_s z\right)\right|$$

Envelope oscillations drive Emittance oscillations



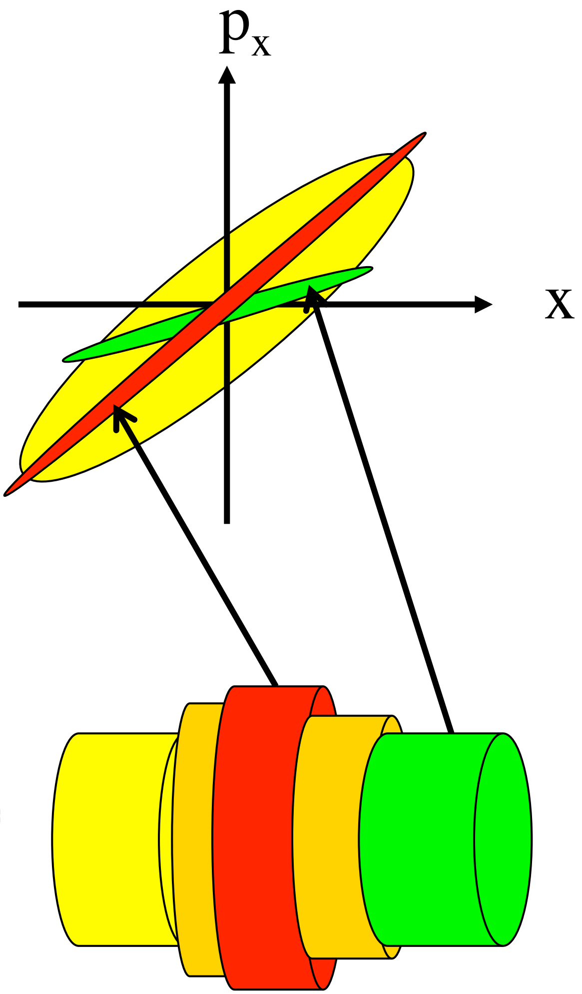
$$\frac{\delta\gamma}{\gamma} = 0$$

$$\sigma' = 0$$

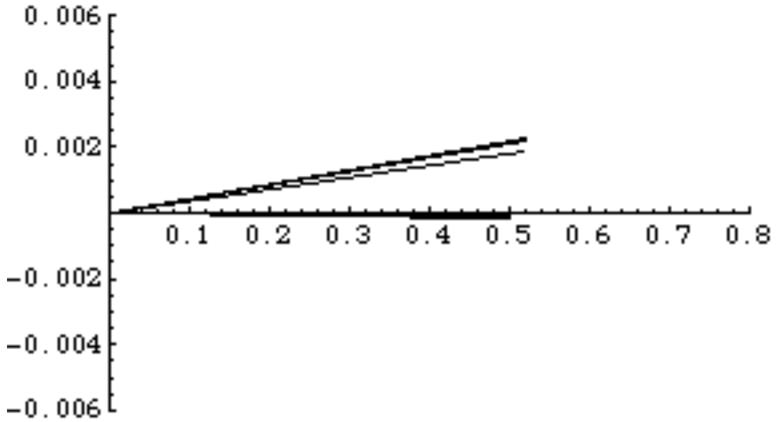
$$\varepsilon_{rms} = \sqrt{\sigma_x^2\sigma_{x'}^2 - \sigma_{xx'}{}^2} = \sqrt{\left(\left\langle x^2\right\rangle\left\langle x'^2\right\rangle - \left\langle xx'\right\rangle^2\right)} \approx \left|sin\left(\sqrt{2}k_s z\right)\right|$$

Emittance Oscillations are driven by space charge differential defocusing in core and tails of the beam

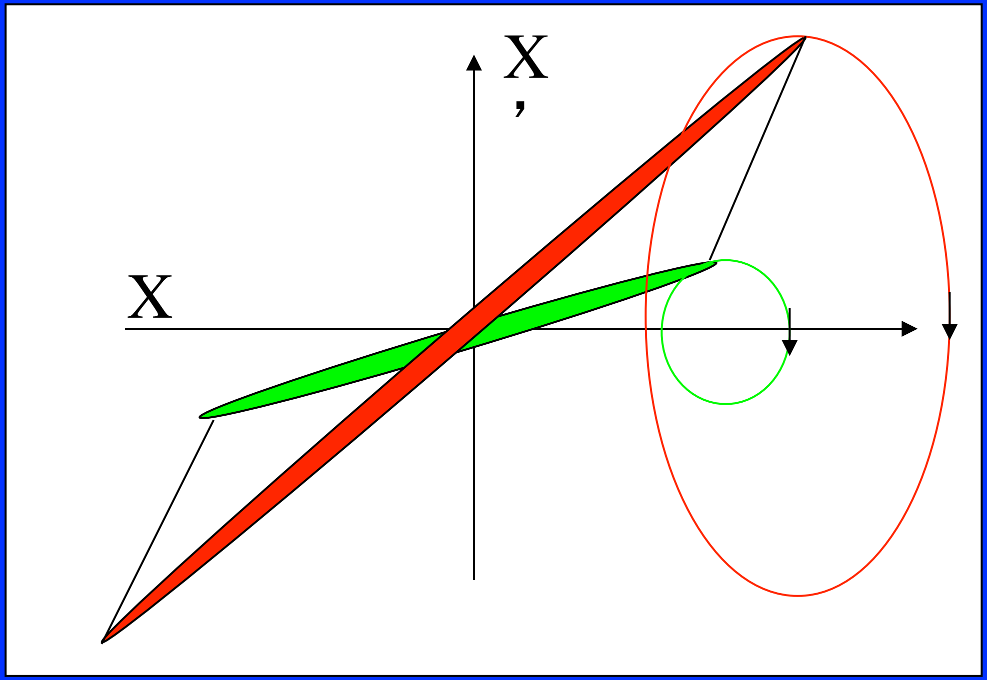
Projected Phase Space



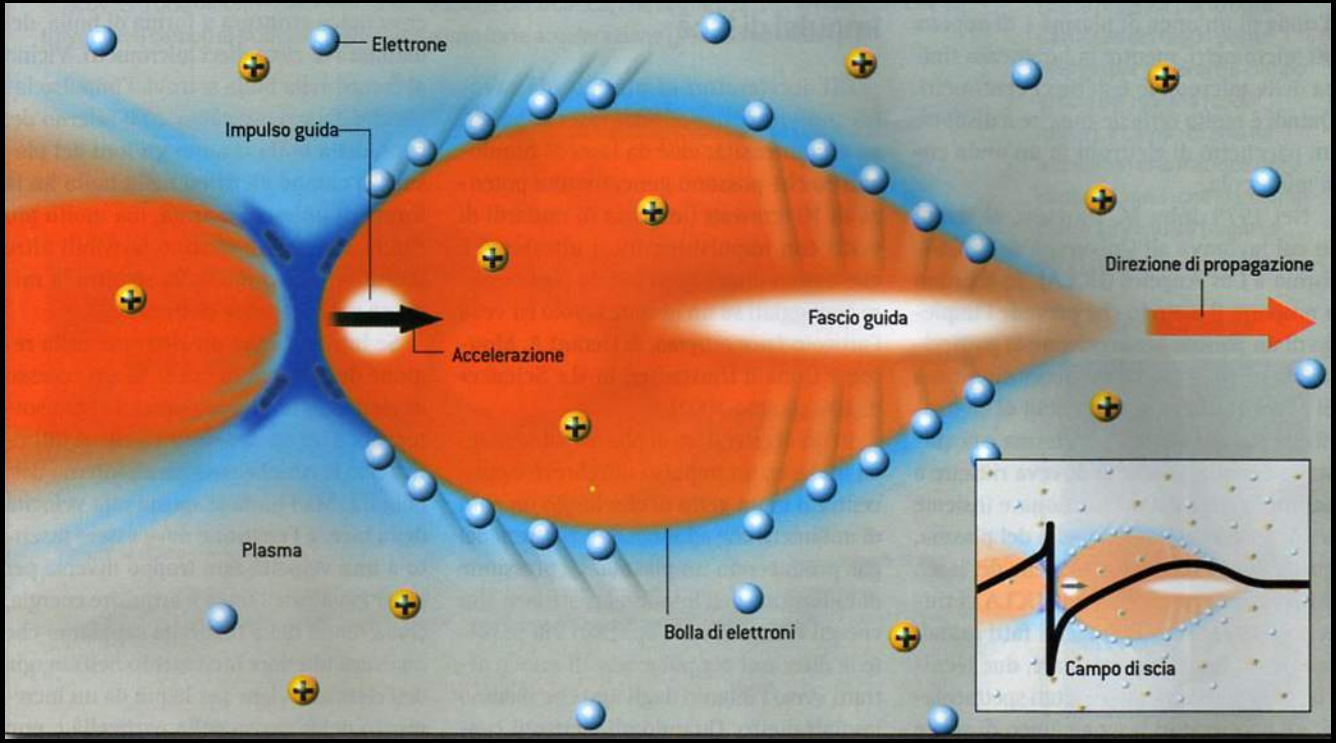
Slice Phase Spaces



Perturbed trajectories oscillate around the equilibrium with the same frequency but with different amplitudes



Plasma Accelerator



Envelope Equation with Longitudinal Acceleration

$$\boxed{\begin{aligned} &p_o = \gamma_o m_o \beta_o c \\ &p_x << p_o \\ &p = p_o + p'z \\ &p' = \left(\beta\gamma\right)' m_o c \end{aligned}}$$

$$\frac{dp_x}{dt} = \frac{d}{dt}\left(px'\right) = \beta c \frac{d}{dz}\left(px'\right) = 0$$

$$x'' + \frac{p'}{p} x' = 0$$

$$\boxed{x'' = -\frac{\left(\beta\gamma\right)'}{\beta\gamma} x'}$$

Envelope Equation with Longitudinal Acceleration

$$\langle xx''\rangle = -\frac{(\beta\gamma)'}{\beta\gamma}\langle xx'\rangle = -\frac{(\beta\gamma)'}{\beta\gamma}\sigma_{xx'}$$

$$\boxed{\sigma_x'' = \frac{\varepsilon_{rms}^2}{\sigma_x^3} - \frac{\langle xx''\rangle}{\sigma_x}}$$

$$\frac{d\sigma_x}{dz} = \sigma_x' = \frac{\sigma_{xx'}}{\sigma_x}$$

Space Charge De-focusing Force

$$\boxed{\sigma_x'' + \frac{(\beta\gamma)'}{\beta\gamma}\sigma_x' + k^2\sigma_x = \frac{\varepsilon_n^2}{(\beta\gamma)^2\sigma_x^3} + \frac{k_{sc}}{\sigma_x}}$$

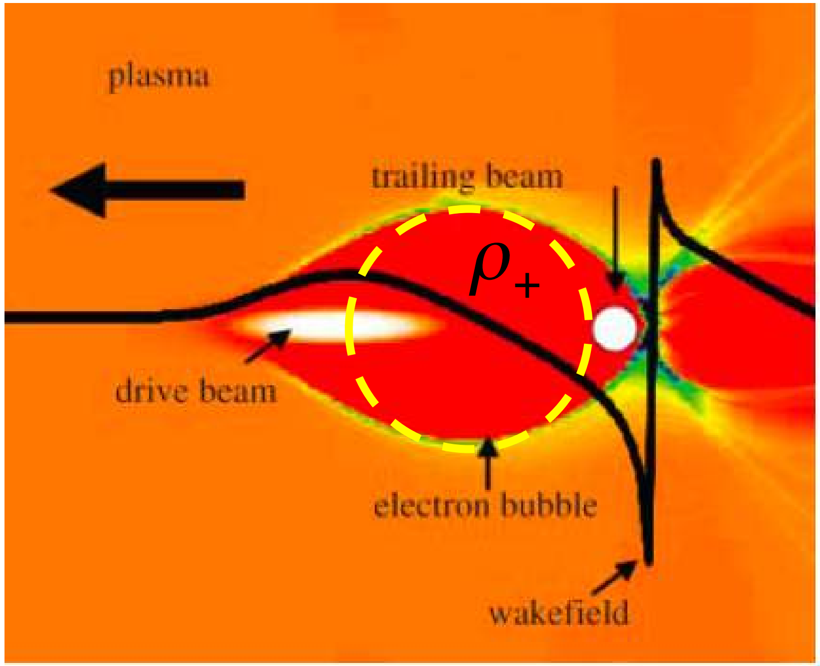
Adiabatic Damping

Emittance Pressure

Other External Focusing Forces

$$\boxed{\varepsilon_n = \beta\gamma\varepsilon_{rms}}$$

Envelope equation in a plasma accelerator



$$R_{sphere} \approx \frac{\lambda_p}{2}$$ Bubble radius

$$n_1 \approx n_{drive}$$ Bubble density

$$E_r = \frac{en_1}{3\varepsilon_o} r$$ Radial field

$$F_r = e\left(E_r - \beta c B_\vartheta\right) = \frac{e^2 n_1}{3\varepsilon_0} r$$

$$x'' = \frac{F_x}{\beta c p} = \frac{e^2 n_1 x}{3\varepsilon_o \gamma m c^2} = \frac{k_p^2}{3\gamma} x \qquad k_p^2 = \frac{e^2 n_1}{\varepsilon_o m c^2} \qquad \langle x x'' \rangle = \frac{k_p^2}{\gamma} \langle x^2 \rangle = \frac{k_p^2}{\gamma} \sigma_x^2$$

$$\boxed{\sigma_x'' + \frac{\gamma'}{\gamma} \sigma_x' + \frac{k_p^2}{3\gamma} \sigma_x = \frac{\varepsilon_n^2}{\gamma^2 \sigma_x^3} + \frac{k_{sc}^o}{\gamma^3 \sigma_x}}$$

When $\eta = \frac{4\gamma k_p^2}{3\gamma'^2} >> 1$ $\rho = \frac{k_{sc}^0 \sigma_x^2}{\gamma_o \varepsilon_n^2} << 1$

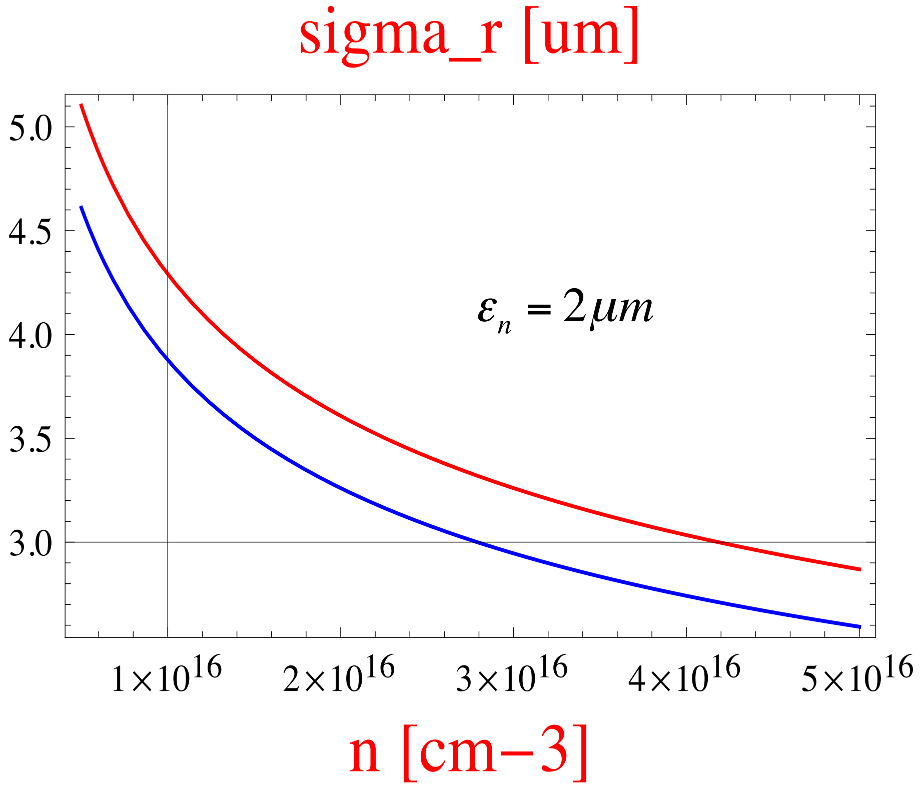
$$\boxed{\gamma'' = 0 \qquad \gamma' \neq 0}$$

$$\boxed{\sigma_x'' + \frac{k_p^2}{3\gamma}\sigma_x = \frac{\varepsilon_n^2}{\gamma^2 \sigma_x^3}}$$

Looking for an equilibrium solution of the form: $\sigma_\varepsilon = \gamma^n \sigma_o$

We get the matching condition with acceleration:

$$\boxed{\sigma_\varepsilon = \sqrt[4]{\frac{3}{\gamma}}\sqrt{\frac{\varepsilon_n}{k_p}}}$$

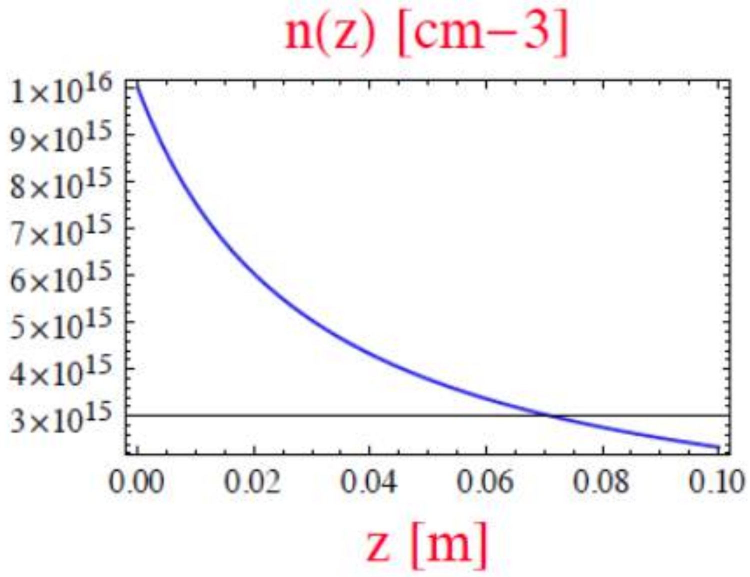


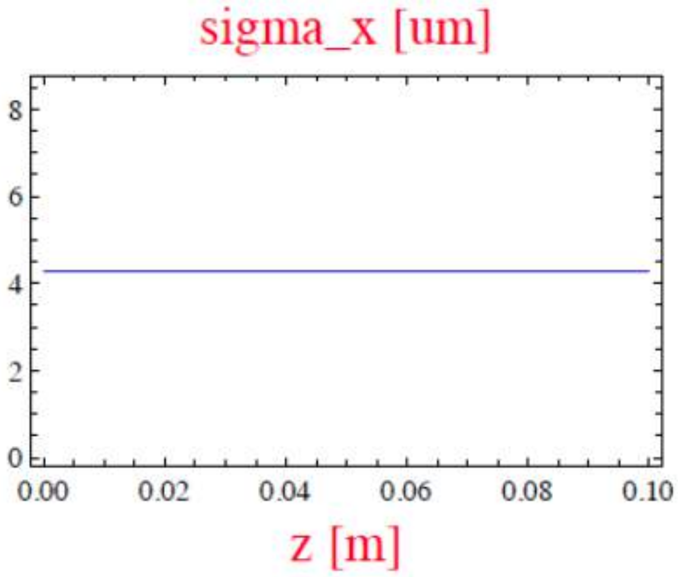
It is an interesting exercise to see the effect of a plasma density vanishing as $n(z)=\frac{\gamma_o}{\gamma(z)}n_o$, giving $k_p^2=\frac{e^2 n_o}{\varepsilon_o mc^2}\frac{\gamma_o}{\gamma}=\frac{\gamma_o}{\gamma}k_{o,p}^2$. In this case the envelope equation

$$\sigma''_x+\frac{\gamma'}{\gamma}\sigma'_x+\frac{\gamma_o k_{o,p}^2}{3\gamma^2}\sigma_x=\frac{\varepsilon_n^2}{\gamma^2\sigma_x^3}$$

admits a constant equilibrium solution:

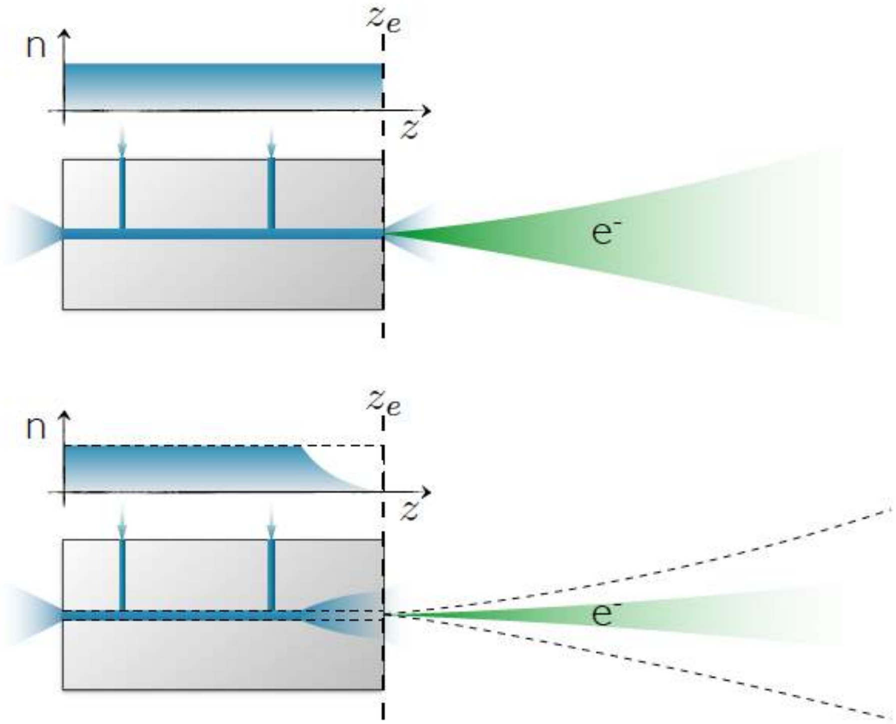
$$\sigma_x=\sqrt[4]{\frac{3}{\gamma_o}}\sqrt{\frac{\varepsilon_{n,}}{k_{o,p}}}$$



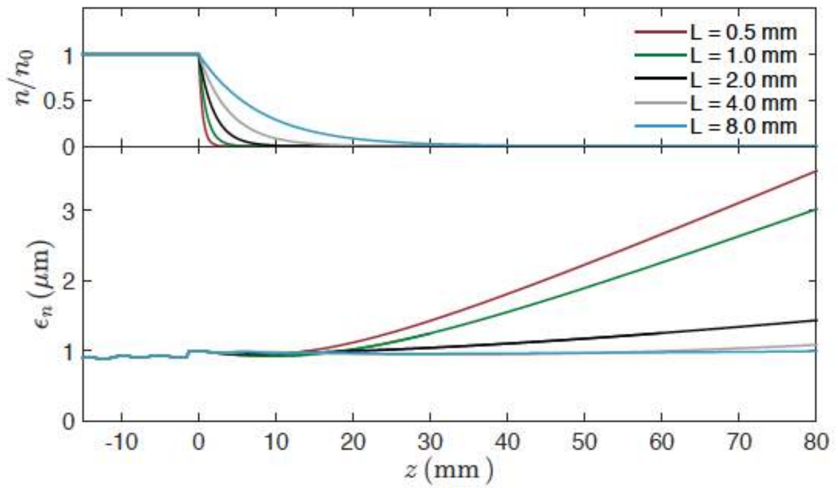


FLASHForward strategy: tailored plasma-to-vacuum transition to adiabatically increase beta, minimize emittance growth

- Concept, theory: *T.Mehrling (FLA), to be published*



Emittance growth depending on transition length



- Plasma-to-vacuum transition » beta for emittance preservation